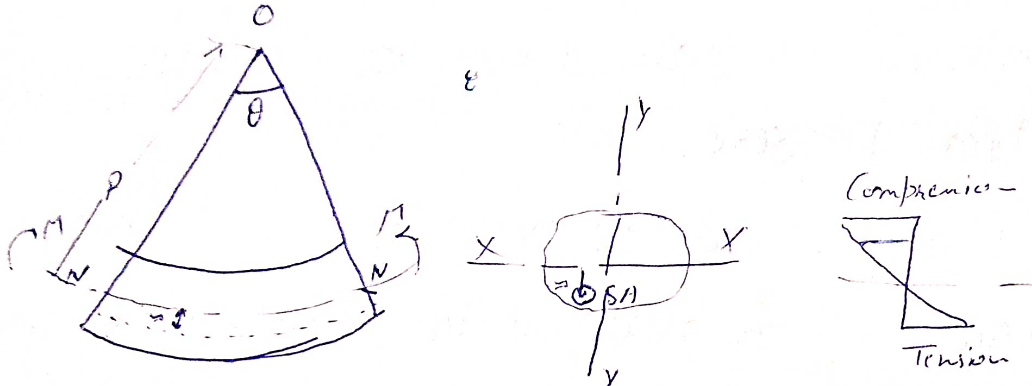
SIMPLE BENDING OF BEAMS

1. Pure Bending - If a length of beam is acted upon by a constant bending moment (zero shearing force), the stresses set up on any cross-section must constitute a pure couple equal in magnitude to the bending moment. Hence it can be deduced that one part in compression & the other part in tension.



From fig it is clear that end sections remain plane & that for initially straight beam the inside or concave edge will be in compression & the outside or convex edge will be in tension. There will be an intermediate surface at which stress is zero ("neutral" surface); the neutral surface cuts any cross-section in the neutral axis.

THEORY OF SIMPLE BENDING :- When a beam is bent due to application of constant bending moment, without being subjected to shear, it is said to be in a state of simple bending. The following assumptions are made in this theory:

1. The material is assumed to be homogeneous, perfectly elastic & isotropic.

2. A transverse section of a beam, which is plane before bending, will remain a plane after bending.

3. The radius of curvature of ~~bending~~ beam before bending is very large in comparison to the transverse section or dimensions of the beam.

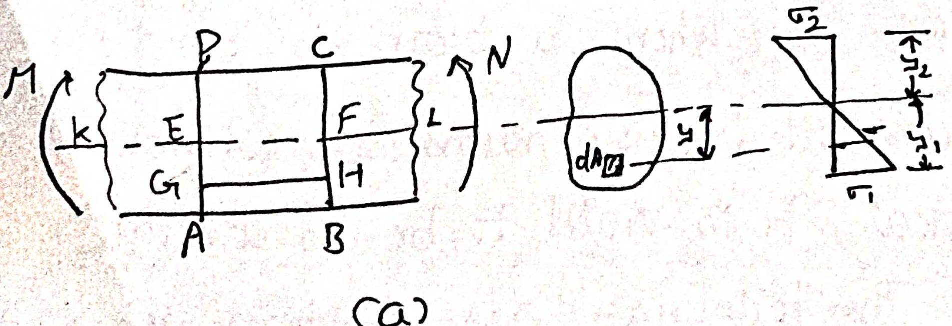
4. The resultant pull or push across a transverse section of beam is zero.

5. The elastic limit is nowhere exceeded.

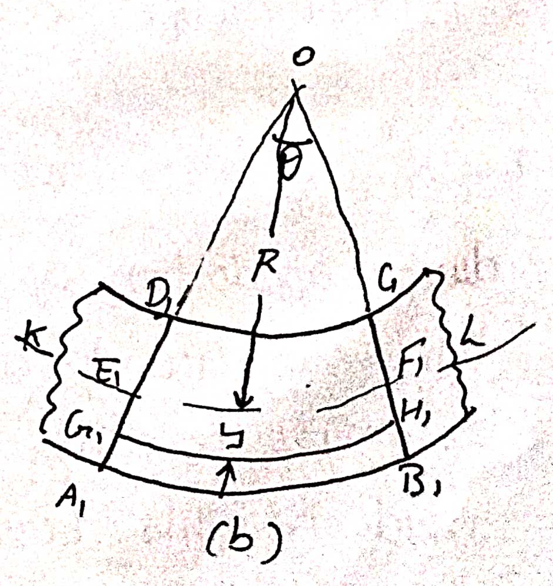
6. Young's modulus of the material is same in tension & compression.

7. The transverse section of beam is symmetrical about an axis passing through the centroid of the section & parallel to the plane of bending.

Consider the portion of a beam subjected to simple bending.



(a)



(b)

In the unstrained state, let GH be a portion of a fibre at a distance y from the centroidal axis KL, its length being determined by the two transverse parallel planes AD & BC. After bending, the planes AD & BC assume the positions A_1D_1 & B_1C_1 respectively, being inclined at an angle θ & intersecting at the point O, the centre of curvature. Let R be the radius of the centroidal surface E_1F_1 so that the radius of surface G_1H_1 is $(R+y)$

Now, $\frac{G_1H_1}{EF} = \frac{(R+y)\theta}{R\theta} = \frac{R+y}{R}$

Strain in the fibres at GH is,

$$\epsilon = \frac{G_1H_1 - GH}{GH}$$

$$= \frac{G_1H_1 - EF}{EF} = \frac{G_1H_1}{EF} - 1 = \frac{R+y}{R} - 1$$

or $\epsilon = \frac{y}{R}$ — (1)

If σ = Intensity of stress in the fibre

$\sigma = E\epsilon$

or $\sigma = \frac{Ey}{R}$ {from (1)}

or $\boxed{\frac{\sigma}{y} = \frac{E}{R}}$

$\sigma \propto y \quad \because \frac{E}{R}$ is constant

$\therefore \sigma = ky$

The stress in the fibres of a beam at a point in the cross-section is proportional to its distance from the centroidal axis.

Now, Consider an element of area dA at a distance y from the centroidal axis. Total force on the element is then equal to

$$dF = \sigma\, dA$$

But $$\frac{\sigma}{y} = \frac{\sigma_1}{y_1}$$

$$\therefore \quad \sigma = \sigma_1 \frac{y}{y_1}$$

$$\therefore \quad dF = \sigma_1 \frac{y}{y_1}\, dA$$

Now the moment of force acting on the elementary area dA

$$dM = \frac{\sigma_1}{y_1}\, y^2\, dA$$

The total moment of all the forces acting on various elements composing the cross-section forms a couple which is equal to the bending moment M. This total moment is called moment of resistance.

$$\therefore \quad M = \frac{\sigma_1}{y_1} \Sigma y^2\, dA$$

$$= \frac{\sigma_1}{y_1}\, I \qquad \{\Sigma y^2\, dA = I,\ M.O.I.\}$$

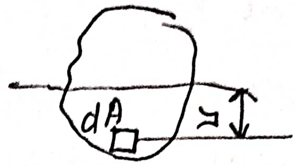
$$\therefore \quad \frac{M}{I} = \frac{\sigma_1}{y_1} = \frac{\sigma}{y}$$

$$\therefore \quad \boxed{\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}}$$ Bending Eqn.

Neutral axis passes through the centroid of section :-

In order to locate the position of the neutral axis, consider an element of area dA at a distance y from centroidal axis.

Total force on the element is then equal to



$$dF = \sigma\, dA$$

But $\frac{\sigma}{y} = \frac{\sigma_1}{y_1}$

$\therefore \quad \sigma = \sigma_1 \frac{y}{y_1}$

& $dF = \sigma_1 \frac{y}{y_1} dA$

Total tensile force on the transverse section below the centroidal axis is then

$$F_1 = \Sigma \sigma_1 \frac{y}{y_1} dA$$
$$= \frac{\sigma_1}{y_1} \Sigma y\, dA$$

If the elementary area is chosen on the upper side, then the total compressive force on the transverse section above the centroidal axis will be,

$$F_2 = \frac{\sigma_2}{y_2} \Sigma y\, dA$$

for equillibrium of beam

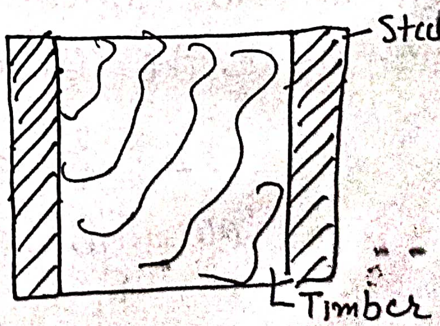
$$F_1 = F_2$$

$$\frac{\sigma_1}{y_1} = \frac{\sigma_2}{y_2}$$

Further, since there is no resultant force across any section, therefore $\Sigma y\, dA = 0$, i.e. the first moment of area about centroidal axis is zero which is to reinforce the beam at regions of high bending moment or to equalise the strength of the beam in tension & compression.

Let us consider a beam made of timber & steel. At the common surfaces, strain in timber & steel is equal

$$\therefore \quad \epsilon_s = \epsilon_t$$



But $\epsilon_s = \dfrac{\sigma_s}{E_s}$

& $\epsilon_t = \dfrac{\sigma_t}{E_t}$

$$\therefore \quad \frac{\sigma_s}{E_s} = \frac{\sigma_t}{E_t}$$

or $$\frac{\sigma_s}{\sigma_t} = \frac{E_s}{E_t}$$

& $\frac{E_s}{E_t} = m$, modular ratio

$$\therefore \frac{\sigma_s}{\sigma_t} = m \qquad -(1)$$

Let y_s & y_t be the distances of the farthermost fibres of steel & timber respectively from the neutral axis. Since strain is directly proportional to stress & stress is directly proportional to the distance from the neutral axis, hence

$$\frac{\varepsilon_s}{\varepsilon_t} = \frac{y_s}{y_t}$$

or $$\frac{(\sigma_s)_{max}}{E_s} \times \frac{E_t}{(\sigma_t)_{max}} = \frac{y_s}{y_t}$$

or $$\frac{(\sigma_s)_{max}}{(\sigma_t)_{max}} = \frac{y_s}{y_t} \cdot \frac{E_s}{E_t} \qquad -(2)$$

Now $$\frac{M}{I} = \frac{\sigma}{y}$$

$$\therefore \frac{M_s}{I_s} = \frac{\sigma_s}{y_s}$$

$$\Rightarrow M_s = \frac{(\sigma_s)_{max} I_s}{y_s}$$

Similarly $$M_t = \frac{(\sigma_t)_{max} I_t}{y_t}$$

Total Moment of Resistance

$$M = M_s + M_t$$

$$= \frac{(\sigma_s)_{max}}{y_s} \cdot I_s + \frac{(\sigma_t)_{max}}{y_t} \cdot I_t$$

$$= \frac{(\sigma_t)_{max}}{y_t} \cdot \frac{E_s}{E_t} \cdot I_s \cdot \frac{\cancel{y_s}}{\cancel{y_s}} + \frac{(\sigma_t)_{max}}{y_t} I_t \quad \left\{ \because \text{from } (2) \right\}$$

$$M = \frac{(\sigma_t)_{max}}{y_t}\left[\frac{E_s}{E_t} I_s + I_t\right]$$

Similarly $$M = \frac{(\sigma_s)_{max}}{y_s}\left[I_s + \frac{E_t}{E_s} I_t\right]$$

Let $\frac{E_s}{E_t} = m$, modular ratio

$$\therefore \quad M = \frac{(\sigma_t)_{max}}{y_t}\left[m I_s + I_t\right]$$

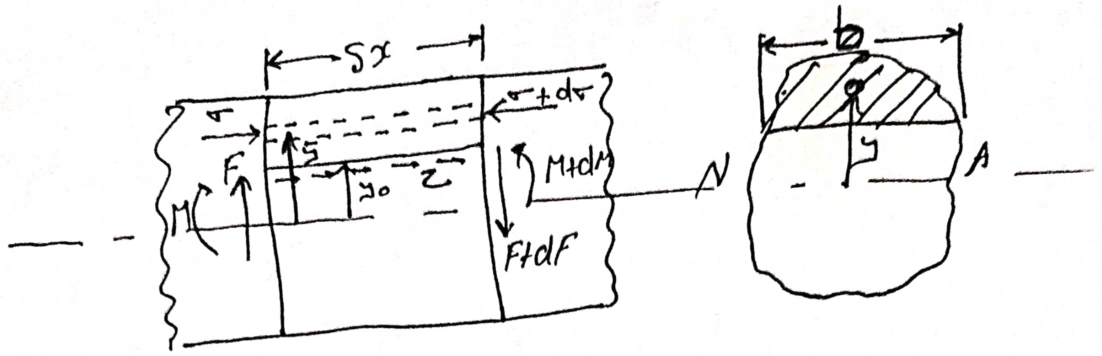
or $$M = \frac{(\sigma_s)_{max}}{y_s}\left[I_s + I_t/m\right]$$

$(I_t + m I_s)$ is called the equivalent moment of inertia of section as if the beam were made of timber. Similarly $(I_s + \frac{I_t}{m})$ is the equivalent moment of inertia of section as if the beam were made of steel.

Distribution of Horizontal shear stress in Beams :-

The shearing force at any cross-section of beam will set up a shear stress on transverse sections which in general will vary across the section. In the following analysis, it will be assumed that the stress is uniform across the width (i.e. to the neutral axis), & also that the presence of shear stress does not affect the distribution of bending stress.

Due to shear stress on transverse planes there will be complementary shear stress on longitudinal planes parallel to the neutral axis. In fig, two transverse sections are shown, at a distance δx



apart, the shearing forces & bending moments being F, F+δF & M, M+δM respectively.

Let τ be the value of complementary shear stress (& hence the transverse shear stress) at a distance y_0 from the neutral axis, $\bar{y}$ is the distance of centroid of A from the neutral axis & b is the width of cross-section at this section.

If σ, $\sigma + d\sigma$ are the normal stresses on an element of area δA at the two transverse sections, then there is a difference of longitudinal force equal to, $dF = d\sigma\, dA$

Now, $F = \sigma\, dA$

& $\sigma = \frac{M}{I} y$

$\therefore F = \frac{M}{I} y\, dA$ — ①

Similarly

$F + dF = \frac{M + dM}{I} y\, dA$ ②

Unbalanced force, $dF = \frac{dM}{I} y\, dA$ {∵ ② − ①}

Total unbalanced force $\Sigma dF = \frac{dM}{I} \Sigma y\, dA$

$F = \frac{dM}{I} \bar{y} A$ ③ {∵ $\Sigma y\, dA = \bar{y} A$}

Now, Due to shear stress τ,

Shear force $= \tau . b . \delta x$ ④

By equation (3) & (4)

$$\tau \cdot b \cdot \delta x = \frac{dM}{I} \cdot \bar{y} A$$

$$\therefore \quad \tau = \frac{dM}{\delta x} \frac{\bar{y} A}{I b}$$

$$\boxed{\tau = \frac{F \bar{y} A}{I b}} \quad \left\{ \because \frac{dM}{dx} = F \right\}$$

1-6, 43, 44, 45