SPRINGS

Helical Spring : A helical spring is a piece of wire coiled in the form of helix. If the slope of the helix of coil is so small that the bending effects can be neglected, then the spring is called a close-coiled spring. In such a spring only torsional shear stresses are introduced.

Open Coiled Spring :- If the slope of helix of coil is quite appreciable, then both bending as well as torsional shear stresses are introduced in the spring, then it is called open-coiled spring.

Spring Index :- It is the ratio of the mean coil diameter to the diameter of the spring. It is denoted by C. C

Stiffness - It is defined as load per unit deflection & is denoted by K.

Torsional rigidity :- Torsional rigidity or torsional stiffness is defined as the torque per unit angular twist.

Helix angle - It is the angle which the axis of spring wire makes with horizontal line perpendicular to the axis of spring.

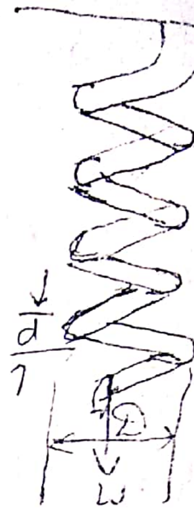
Close - Coiled Helical Springs

Analysis → Consider a close coiled helical spring under the action of axial load.

W : axial load

D : mean coil diameter

d = diameter of spring wire
δ = Axial Deflection
G = Modulus of rigidity
θ = Angular deflection
n = number of coils
τ = Maximum shear stress induced



Since the angle of helix is small the action on any cross-section is approximately a bare torque = $W.D/2$ & the bending & shear effects may be neglected.

& Total length of wire $l = \pi D n$

Now torque equation

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{l}$$

$$\frac{W.D/2}{\frac{\pi}{32}d^4} = \frac{\tau}{d/2} = \frac{G\theta}{\pi D n}$$

$$\tau = \frac{WD}{2} \times \frac{d}{2} \times \frac{32}{\pi d^4} = \boxed{\frac{8WD}{\pi d^3}}$$

$$\tau = \frac{8WD}{\pi d^3}$$

Max shear stress $\boxed{\tau = \frac{8WD}{\pi d^3}}$

$$\& \ \theta = \frac{Tl}{GJ} = \frac{WD}{2} \times \frac{\pi D n}{G \frac{\pi d^4}{32}} =$$

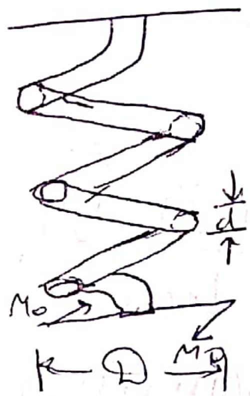
$$\boxed{\theta = \frac{16WD^2 n}{Gd^4}}$$

Deflection, $\delta = \frac{\theta D}{2}$

$$\therefore \boxed{\delta = \frac{8WD^3 n}{Gd^4}}$$

Stiffness $k = \frac{W}{\delta} = \frac{Gd^4}{8D^3 n}$

Axial Torque: Consider a closely coiled helical spring subjected to an axial couple M_0.



The couple M_0 produces bending in the coils of the spring. Due to the couple the coils curvature increases or decreases.

Let R_1 = Initial radius of curvature

R_2 = Changed radius of curvature

n_1, n_2 = Initial & changed number of coils respectively.

Change in curvature

$$= \frac{1}{R_2} - \frac{1}{R_1} = \frac{M_0}{EI} \quad -(1) \quad \left\{ \because \frac{M}{I} = \frac{E}{R} \right\}$$

Now $l = 2\pi R_1 n_1 = 2\pi R_2 n_2$

$$\therefore \frac{1}{R_2} - \frac{1}{R_1} = \frac{2\pi}{l}(n_2 - n_1) \quad -(2)$$

$$\therefore \boxed{M_0 = \frac{2\pi EI}{l}(n_2 - n_1)} \quad \{\because \text{From (1) \& (2)}\}$$

Angle of twist, $\phi = 2\pi(n_2 - n_1)$

& also $\phi = \frac{M_0 l}{EI} = \frac{2\pi R_1 n_1 M_0}{EI}$

For spring of round wire of diameter d, number of coils n, & mean coil diameter $D = 2R$,

$$\phi = \frac{64\pi D n M_o}{E\pi d^4}$$

$$\therefore \boxed{\phi = \frac{64 D n M_o}{E d^4}}$$

& bending stress $\sigma_b = \frac{M_o\, d/2}{\frac{\pi d^4}{64}}$

$$\boxed{\sigma_b = \frac{32 M_o}{\pi d^3}}$$

Strain Energy in the Spring:-

(a) Axial Load

Strain energy, $U = \frac{1}{2} T \times \theta$

$$= \frac{1}{2} \times \frac{WD}{2} \times \frac{16 W D^2 n}{G d^4} = \frac{4W^2 D^3 n}{G d^4}$$

Now $\tau = \frac{8WD}{\pi d^3}$

$$\therefore W = \frac{\pi d^3 \tau}{8D}$$

$$\therefore U = \left(\frac{4D^3 n}{G d^4}\right)\left(\frac{\pi d^3 \tau}{8D}\right)^2$$

$$= \frac{\tau^2}{G} \times \frac{1}{16} \times D \times n \times d^2 \times \pi^2$$

$$= \frac{\tau^2}{4G} (\pi D n)\left(\frac{\pi}{4} d^2\right)$$

$$\boxed{U = \frac{\tau^2}{4G} \times \text{Vol. of the spring}}$$

(b) Axial twist.

Strain energy, $U = \frac{1}{2} M_0 \phi$

$= \frac{1}{2} M_0 \frac{64 M_0 D n}{E d^4}$

$= \frac{32 M_0^2 D n}{E d^4}$

Now $M_0 = \frac{\pi d^3}{32} \sigma_b$

$U = \frac{32 D n}{E d^4} \left(\frac{\pi d^3}{32} \sigma_b \right)^2$

$= \frac{\sigma_b^2}{8E} \left(\frac{\pi d^2}{4} \right) \times \pi D n$

$$\boxed{U = \frac{\sigma_b^2}{8E} \times \text{Vol.}}$$

Springs in series:- When two springs of different stiffness are joined end to end carry a common load W, they are said to be connected in series.

Total Deflection $\frac{W}{K} = \frac{W}{K_1} + \frac{W}{K_2}$

$$\boxed{\frac{1}{K} = \frac{1}{K_1} + \frac{1}{K_2}}$$

Spring in ~~series~~ Parallel:- When two springs are joined in such a way that they have a common deflection, then they are said to be ~~been~~ connected in parallel.

Total load; $W = W_1 + W_2$

Deflection $\delta = \frac{W}{K} = \frac{W_1}{K_1} = \frac{W_2}{K_2}$

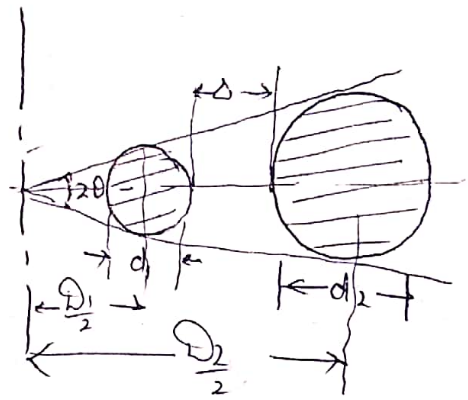
$W_1 = W\frac{K_1}{K}$, $W_2 = W\frac{K_2}{K}$

$\therefore W = \frac{W}{K}(K_1 + K_2)$

$$\boxed{K = K_1 + K_2}$$

Concentric or Cluster Springs :-

Concentric or cluster springs are close-coiled helical springs placed one inside the other.



Radial clearance

$$\Delta = \left(\frac{D_2 - D_1}{2}\right) - \left(\frac{d_1 + d_2}{2}\right)$$

The following conditions have to be satisfied by springs.

First Condition :- When the springs are of equal free length & made of same material, then the max. shear stress in the springs is equal.

$$\frac{8W_1D_1}{\pi d_1^3} = \frac{8W_2D_2}{\pi d_2^3}$$

$$\therefore \frac{W_1}{W_2} = \frac{D_2}{D_1}\left(\frac{d_1}{d_2}\right)^3 \quad \text{—(1)}$$

Second Condition :- The deflection is same for both the springs

$$n_1 d_1 = n_2 d_2$$

$$\therefore \frac{n_2}{n_1} = \frac{d_1}{d_2} \quad \text{—(2)}$$

Third Condition:- The Deflection is same for both the springs.

$$\frac{8W_1D_1^3n_1}{G_1d_1^4} = \frac{8W_2D_2^3n_2}{G_2d_2^4}$$

$$\therefore \frac{W_1}{W_2} = \left(\frac{D_2}{D_1}\right)^3\left(\frac{d_1}{d_2}\right)^4\left(\frac{G_1}{G_2}\right)\left(\frac{n_2}{n_1}\right) \quad (3)$$

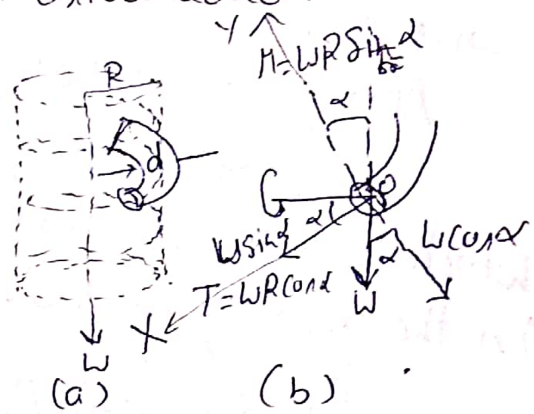
& from fig $\tan\theta = \frac{d_1}{D_1} = \frac{d_2}{D_2}$

Also. $W_1 + W_2 = W$

Open Coiled Helical Spring

Axial Load

(a) Axial Deflection:- Consider an open coiled helical spring under the action of an axial load.



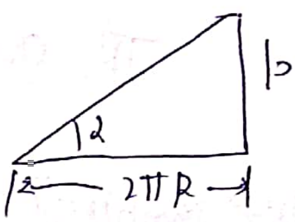
Let α = Constant helix angle which the coil makes with the planes perpendicular to the axis of spring.

$D = 2R$ = Mean Diameter of the spring coil.

n = number of open coils

p = pitch of coils

W = Axial Load



Then $\tan\alpha = \frac{p}{2\pi R}$

Length of spring wire,

$$l = \frac{2\pi R n}{\cos\alpha}$$

$$= 2\pi R n \sec\alpha$$

The couple acting on the spring due to load W is

$$= WR$$

& The couple can be resolved in to two components along OX & OY.

Along OX, torque

$$T = WR\cos\alpha$$

which at all sections produces torsion along the wire axis.

Along OY, bending moment

$$M = WR\sin\alpha$$

which at all sections produces bending moment in the spring wire.

Let δ = deflection of the spring under the load.

The average external work done

$$= \frac{1}{2}W\delta$$

Also combined strain energy stored in the spring under bending & torsion is

$$U = \frac{1}{2}T\theta + \frac{1}{2}M\phi$$

Where θ_1 = angle of twist due to twisting moment T

ϕ = Angle subtended by the bent wire at the centre of curvature due to bending moment M.

Now $\theta = \frac{Tl}{GJ}$

$J = \frac{\pi d^4}{32}$

& $\frac{M}{I} = \frac{E}{R}$

Also $\frac{l}{R} = \phi$

$\therefore \frac{M}{I} = \frac{E\phi}{l} \Rightarrow \phi = \frac{Ml}{EI}$

$\therefore U = \frac{1}{2}\frac{T^2 l}{GJ} + \frac{1}{2}\frac{M^2 l}{EI}$

$\frac{1}{2}W\delta = \frac{1}{2}\frac{(WR\cos\alpha)^2 \cdot l}{GJ} + \frac{1}{2}\frac{(WR\sin\alpha)^2 \cdot l}{EI}$

$\delta = \frac{WR^2\cos^2\alpha}{GJ} \cdot l + \frac{WR^2\sin^2\alpha}{EI} \cdot l$

$= \left[\frac{WR^2\cos^2\alpha}{\frac{\pi}{32}d^4 \times G} + \frac{WR^2\sin^2\alpha}{\frac{\pi d^4}{64} \times E}\right] 2\pi Rn\sec\alpha$

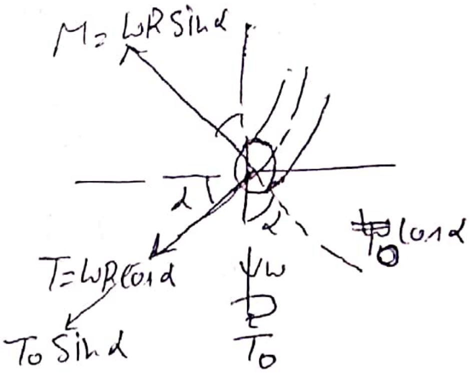
$$\boxed{\delta = \frac{64WR^3 n\sec\alpha}{d^4}\left[\frac{\cos^2\alpha}{G} + \frac{2\sin^2\alpha}{E}\right]}$$

For a close coiled helical spring $\alpha = 0$

$$\delta = \frac{64 W R^3 n}{G d^4}$$

(b) Angular rotation:- On account of M there is bending in the plane of the axis of the wire & due to T, there is twisting in the wire.

Now, consider an imaginary axial torque T_0 applied to the spring, together with W producing an angular rotation β of one end of the spring relative to the other. Then combined twisting moment on the spring is



$$T' = WR\cos\alpha + T_0\sin\alpha$$

& the combined bending moment is

$$M' = T_0\cos\alpha - WR\sin\alpha$$

Total strain energy of the system is then

$$U = \frac{1}{2}\frac{T'^2 l}{GJ} + \frac{1}{2}\frac{M'^2 l}{EI}$$

$$= \frac{1}{2}\frac{[WR\cos\alpha + T_0\sin\alpha)^2 \cdot l}{GJ} + \frac{1}{2}\frac{(T_0\cos\alpha - WR\sin\alpha)^2 l}{EI}$$

Using Castigliano's theorem, β is given by $\left(\frac{dU}{dT_0}\right)$ when $T_0 = 0$

$$\therefore \beta = \frac{\partial U}{\partial T_0} = \frac{1}{2}\left(\frac{2T_0\sin^2\alpha + 2WR\sin\alpha\cos\alpha}{GJ}\right) l + \frac{1}{2}\frac{(2T_0\cos^2 R - 2WR\sin\alpha\cos\alpha)}{EI} \cdot l$$

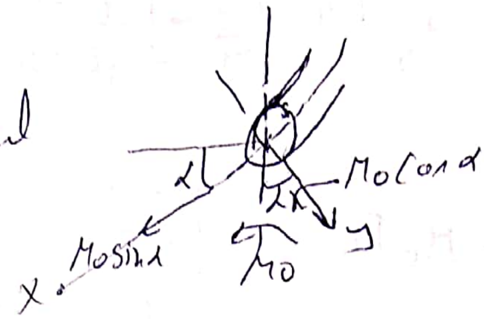
$T_0 = 0$

$$\therefore \beta = \frac{2WRl\cos\alpha\sin\alpha}{2GJ} - \frac{2WRl\sin\alpha\cos\alpha}{2EI}$$

$$\therefore \boxed{\beta = \frac{64 W R^2 L \sin\alpha}{d^4}\left[\frac{1}{G} - \frac{2}{E}\right]}$$

AXIAL TORQUE :-

(a) Axial Rotation:-



Consider an open coiled helical spring under the action of axial torque M_0. The axial torque M_0 can be split in to two components

Component along OX

$$T = M_0 \sin\alpha$$

which at all sections produces torsion in the spring wire.

Component along OY,

$$M = M_0 \cos\alpha$$

which at all sections produces bending moment in the spring wire & tends to change the curvature of the coils.

Let β = angle through which the free end rotates

Work done by applied torque

$$= \frac{1}{2} M_0 \beta$$

Strain energy stored in the spring

$$U = \frac{1}{2} T\theta + \frac{1}{2} M\phi$$

From above two equations, we get

$$\frac{1}{2} M_0 \beta = \frac{1}{2} T\theta + \frac{1}{2} M\phi$$

$$\frac{1}{2} M_0 \beta = \frac{1}{2}\frac{T^2 l}{GJ} + \frac{1}{2}\frac{M^2 l}{EI}$$

$$M_0 \beta = \left[\frac{M_0^2 \sin^2\alpha}{G \times \frac{\pi d^4}{32}} + \frac{M_0^2 \cos^2\alpha}{E\frac{\pi d^4}{64}}\right] 2\pi R n \sec\alpha$$

$$\beta = \frac{64 M_0 R n \sec\alpha}{d^4}\left[\frac{\sin^2\alpha}{G} + \frac{2\cos^2\alpha}{E}\right]$$

If $\alpha \simeq 0$, then

$$\beta = \frac{64 M_0 R n}{d^4}\left[0 + \frac{2}{E}\right]$$

$$\boxed{\beta = 128\frac{M_0 R n}{E d^4}}$$

(b) Axial Deflection:- Assuming an imaginary axial load W_0 applied to the spring, the total strain energy becomes

when $T' = W_0 R\cos\alpha + M_0 \sin\alpha$

& $M' = M_0\cos\alpha - W_0 R\sin\alpha$

$$\therefore U = \frac{l(W_0 R\cos\alpha + M_0\sin\alpha)^2}{2GJ} + \frac{(M_0\cos\alpha - W_0 R\sin\alpha)^2 l}{2EI}$$

Using Castigliano's theorem

$$\delta = \frac{\partial U}{\partial W_0}$$

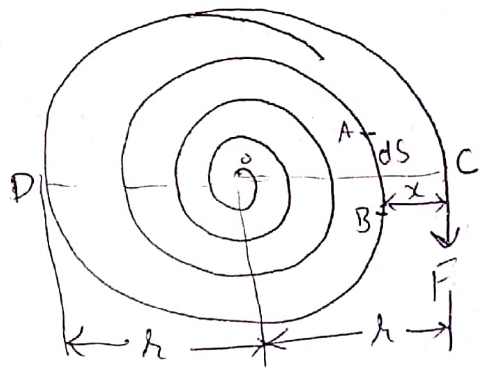
When $w_0 = 0$

$$\therefore \delta = M_0 R l \cos\alpha \sin\alpha \left[\frac{1}{GJ} - \frac{1}{EI}\right]$$

$$\boxed{\delta = \frac{64 M_0 R^2 n \sin\alpha}{d^4}\left[\frac{1}{G} - \frac{2}{E}\right]}$$

FLAT SPIRAL SPRING :-

Consider a flat spiral spring. In such a



spring one end of the spring is attached to a winding spindle at O & the other end is fixed at a pin C. Suppose a couple M is applied to the spindle, then there will be reaction F at C.

Consider a small portion AB = ds of the spring wire. Then the bending moment on AB

$$M = F \times x$$

Let $d\theta$ = Change in angle between two ends of AB before & after straining.

Then the change in curvature of AB

$$= \frac{d\theta}{ds}$$

But change in curvature $= \frac{\text{Bending moment}}{EI}$

$$\frac{d\theta}{ds} = \frac{M}{EI} \qquad \left\{ \frac{M}{I} = \frac{E}{R} \right\}$$

$$d\theta = \frac{M\,ds}{EI}$$

$$= \frac{F x\,ds}{EI}$$

Total angle of winding up due to straining becomes,

$$\int d\theta = \int \frac{F.x\,ds}{EI}$$

$$\theta = \int \frac{F}{EI} . x\,ds$$

But $\int x.ds$ = Moment of the profile of the spring about C

= Length of spring × Distance of the winding spindle i.e. point O from CE

$= l \times r$

$$\boxed{\therefore \theta = \frac{Flr}{EI} = Fr\left(\frac{l}{EI}\right)}$$

But $Fr = M$

$$\therefore \theta = \frac{Ml}{EI}$$

Energy stored in the spring

$$U = \frac{1}{2} M\theta$$

$$= \frac{1}{2} Fr \cdot \frac{Frl}{EI}$$

$$\boxed{U = \frac{1}{2} \frac{F^2 r^2 l}{EI}}$$

If b = width of cross-section of spring wire

d = depth of spring wire

Then the bending stress in the spring wire AB is

$$\frac{\sigma}{y} = \frac{M}{I}$$

$y = d/2$

$$\boxed{\sigma = \frac{Md}{2I} = \frac{F \times d}{2I}}$$

The maximum stress occur at point D & A becomes:

$$\sigma_{max} = \frac{F \cdot 2rd}{2I} = \frac{Frd}{I}$$

$$= \frac{Frd}{\frac{bd^3}{12}} = \frac{12 Fr}{bd^2} = \frac{12 M}{bd^2}$$

$$\boxed{\therefore M = \frac{bd^2}{12} \times \sigma_{max}}$$

Max. strain energy stored

$$U_{max} = \frac{1}{2} \frac{M^2 l}{EI} = \frac{1}{2} \frac{\sigma_{max}^2 b^2 d^4}{144} \times \frac{l}{E \frac{bd^3}{12}}$$

$$\boxed{U_{max} = \frac{\sigma_{max}^2}{24E} \times \text{Vol. of spring}}$$

LEAF SPRING :- This type of spring is commonly used in carriages such as cars, lorries & railway wagons. It is made up of number of leaves of equal width & thickness, but varying length, placed in lamination & loaded as a beam.

There are two main types ; the 'semi-elliptic' simply supported at its ends & centrally loaded & the 'quarter elliptic' or cantilever type.

1. Semi-Elliptical type

It is assumed for the development of a simplified theory that the ends of each leaf - where it extends beyond its neighbour - are tapered uniformly to a point; also, in order to complete the set, the shortest leaf should be a diamond shape.

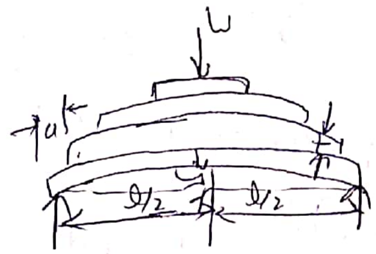
l = Span

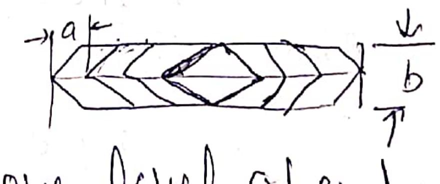
b = width of leaves

t = thickness

W = Central load

y = rise of crown above level of ends



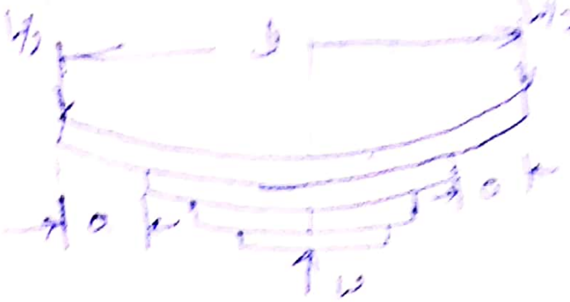


Friction between the plates being neglected, each leaf is free to slide over its neighbours, & since they all maintain the same radius of curvature they can be imagined to be arranged side by side to form a beam of constant depth & varying width.

Leaf Springs

A leaf spring is a beam of uniform strength supported at the centre & loaded at the ends. It consists of a number of overlapping beams, each of the same width & depth but varying in length. Each leaf is shorter than the one above it by a constant, called the overlap. Each plate is free to slide relative to its neighbours & each plate will act as a separate beam. Each plate has initially the same curvature i.e. contact is only at the ends.

Semi-Elliptical Springs



Let W = load acting at mid-point

n = Number of plates

a = Overlap at each end

l = length of spring

b = width of plate

d = depth of plate

$l = 2an$

$\therefore a = \frac{l}{2n}$

Maximum bending moment,

$$M_{max} = \frac{Wl}{4}$$

Bending moment for each plate

$$M = \frac{Wl}{4n}$$

Bending stress, $\sigma = \frac{M}{Z}$

$$= \frac{Wl}{4n} \cdot \frac{6}{bd^2} =$$

$$\boxed{\sigma = \frac{3Wl}{2nbd^2}}$$

Change of curvature for each plate after loading.

$$\frac{1}{R} = \frac{M}{EI} = \frac{Wl}{4n} \times \frac{12}{Ebd^3} = \frac{3Wl}{nEbd^3} \quad -(1)$$

$$\boxed{\frac{1}{R} = \frac{3Wl}{nEbd^3}}$$

Now $EI\frac{d^2y}{dx^2} = M$

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = \frac{3Wl}{nEbd^3} \quad \{\text{from } (1)\}$$

$$y = \int_0^{l/2}\int_0^{l/2} \frac{3Wl}{nEbd^3}\, dx\, dx$$

$$= \frac{3Wl}{nEbd^3} \int_0^{l/2} x\, dx$$

$$= \frac{3Wl}{nEbd^3} \cdot \frac{l^2}{8}$$

Deflection at centre, $\delta = \frac{3}{8}\frac{Wl^3}{nEbd^3}$

Strain energy absorbed

$$U = \frac{M^2}{2EI} \times \text{Length of beam}$$

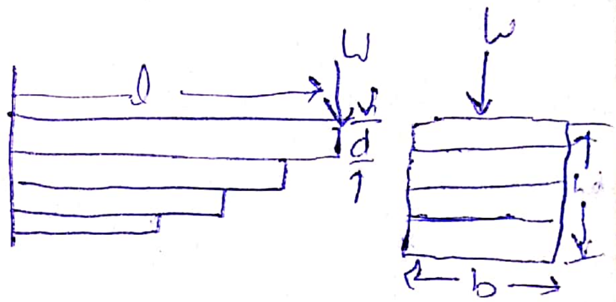
$$= \frac{\left(\frac{bd^2}{6}\sigma\right)^2}{2E\frac{bd^3}{12}} \times \text{Length}$$

$$= \frac{\sigma^2}{6E} \times bd \times \text{total length of all beam}$$

$$\boxed{U = \frac{\sigma^2}{6E} \times \text{Vol. of spring}}$$

QUARTER Elliptic LEAF SPRING: –

Substitute $W = 2W$, & $l = 2l$



$$\sigma = \frac{3}{2}\frac{(2W)(2l)}{nbd^2}$$

$$\boxed{\sigma = \frac{6Wl}{nbd^2}}$$

$$\delta = \frac{3}{8} \cdot \frac{2W(2l)^3}{nEbd^3} = \frac{6Wl^3}{nEbd^3}$$

Strain energy absorbed

$$U = \frac{M^2}{2EI} \times \text{Length of beam}$$

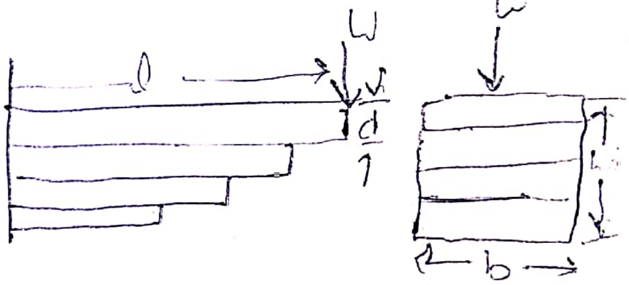
$$= \frac{\left(\frac{bd^2}{6}\sigma\right)^2}{2E\frac{bd^3}{12}} \times \text{length}$$

$$= \frac{\sigma^2}{6E} \times bd \times \text{total length of all beam}$$

$$\boxed{U = \frac{\sigma^2}{6E} \times \text{Vol. of spring}}$$

QUARTER Elliptic LEAF SPRING :-

Substitute $W = 2W$, & $l = 2l$



$$\sigma = \frac{3}{2}\frac{(2w)(2l)}{nbd^2}$$

$$\boxed{\sigma = \frac{6Wl}{nbd^2}}$$

$$\delta = \frac{3}{8} \cdot \frac{2w(2l)^3}{nEbd^3} = \frac{6Wl^3}{nEbd^3}$$