UNIT STRAIN ENERGY & IMPACT LOADING

Whenever a body is strained, the energy is absorbed in the body. The energy, which is absorbed in the body due to straining effect is known as strain energy. The strain energy stored in the body is equal to work done by the applied load in straining the body. The load applied may be gradually or suddenly or with an impact.

DEFINITIONS :-

* Resilience :- The strain energy stored in a body is known as resilience.
* Strain Energy :- The work done in straining the material, with in elastic limits, is known as strain energy.
* Proof Load :- The maximum load which can be applied to a member without its being permanently deformed is called proof load.
* Proof Resilience :- The strain energy stored in the body will be maximum when the body is strained up to elastic limit. Hence the proof resilience is the quantity of strain energy stored in a body when strained up to elastic limit.
* Modulus of resilience :- It is defined as the proof resilience of a material per unit volume.

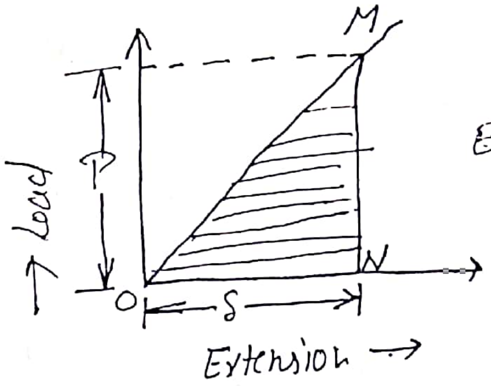
$$\text{Modulus of resilience} = \frac{\text{Proof resilience}}{\text{Vol. of the body}}$$

Expression for Strain Energy stored in a body, when the load is applied gradually. :-

(a) Consider a bar of length L, area of cross-section A, subjected to an axial load P under tensile test upto elastic limit. The tensile load P increased gradually from zero to a value of P & the extension of body increases from zero to the value of δ.

Let δ = Extension of the bar

E = Modulus of elasticity



$E = \frac{\text{Stress}}{\text{Strain}}$

$E = \frac{Pl}{A \cdot \delta}$

$\delta = \frac{Pl}{AE}$

Then, $\delta = \frac{PL}{AE} = \frac{\sigma L}{E}$ ... (1)

where, $\sigma = P/A$

Work done in straining the bar = Area of Load extension curve

$$W = \frac{1}{2} \times P \times \delta$$

$$= \frac{1}{2} \times P \times \frac{Pl}{AE} \quad \{\because \text{From } (1)\}$$

$$= \frac{1}{2} \frac{P^2 L}{AE} = \frac{1}{2} \frac{\sigma^2}{E} \times A \times L$$

$$= \frac{1}{2} \times \frac{\sigma^2}{E} \times V$$

But work done in straining the bar = Strain energy U stored in the bar

$$U = \frac{1}{2} \times \frac{\sigma^2}{E} \times V$$

& Modulus of resilience = $\frac{\text{Total strain energy}}{\text{Volume}} = \frac{\sigma^2}{2E}$

Q. Proof resilience $= \frac{1}{2} \frac{\sigma_{PR}^2}{E} \times V$

where, σ_{PR} = Stress at elastic limit

Also,
$\sigma = E\epsilon$ where ϵ = strain

$$\therefore U = \frac{1}{2}\frac{\sigma^2}{E} \times V$$

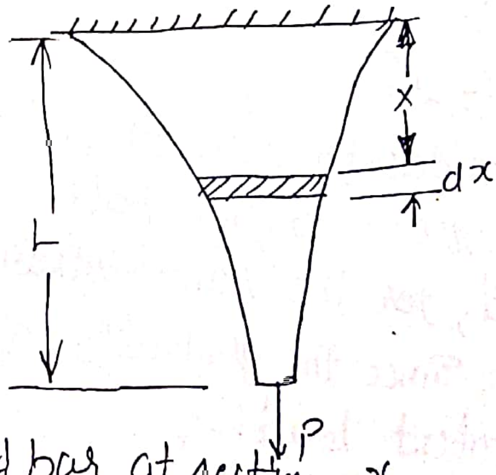
$$= \frac{1}{2}\frac{E^2\epsilon^2}{E} \times V$$

$$= \frac{1}{2}\sigma \times \epsilon \times V$$

$U = \frac{1}{2} \times$ Stress $\times$ strain $\times$ Volume

(b) Now, we consider the case when the area of cross-section of bar, is not uniform

Consider a section at a distance x from the top of the bar.
Let dx be the length of bar between x & $x+dx$.



$$\therefore dU = \frac{1}{2}\frac{\sigma^2}{E} \times A\,dx$$

where A is area of cross-section of bar at section x

Now $\sigma = \frac{P}{A}$

$$dU = \frac{1}{2}\frac{P^2}{AE}dx$$

Total energy, $\int dU = \frac{1}{2}\int_0^L \frac{P^2}{AE}dx$

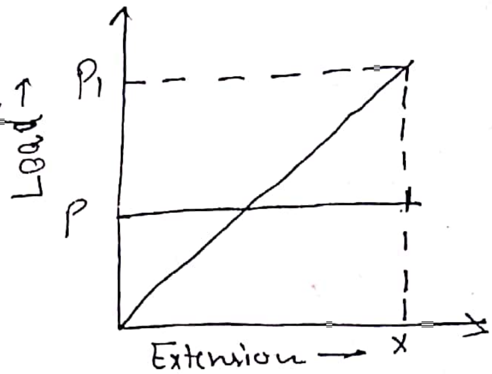
$$\boxed{U = \frac{1}{2}\frac{P^2}{E}\int_0^L \frac{dx}{A}}$$

Strain Energy due to Suddenly applied load :-

When the load is applied suddenly to a body, the load is constant throughout the process of the deformation of the body.

Let a tensile load P be applied to a material suddenly & produce an extension x. The work done will be given by Px. Suppose P_1 to be a load which increases gradually from zero & produces the same extension x, the workdone now is given by $\frac{1}{2} P_1 x$

Since the bar is strained an equal amount in each case & consequently the applied energies will be equal.



$$\therefore \frac{1}{2} P_1 x = Px$$

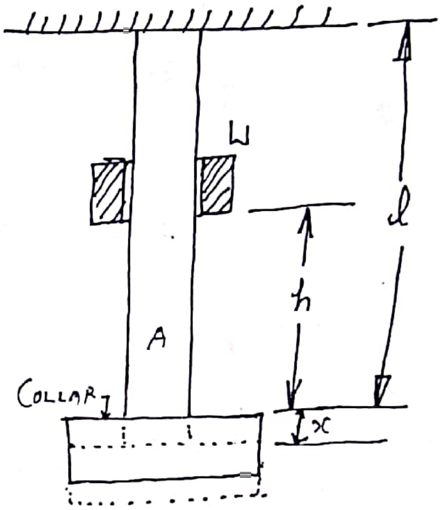
$$\therefore P_1 = 2P$$

Thus a suddenly applied load required to produce a given strain is just half the magnitude of that required, for the same strain, when gradually applied. Since the gradually applied load is called an equivalent load.

Strain Energy due to impact or shock load :-

Let the weight W fall freely on to a collar or anvil carried at the lower end of a bar of area of cross-section A, length l, fixed at the upper end through the height h. The following assumptions are made.

(i) The weight W falls freely
(ii) The support is rigid
(iii) There is no loss of energy on striking the anvil or collar due to rebound.



x = extension of the bar
σ = Stress induced in the bar

Potential energy lost by the weight
= Strain energy stored in the bar

$$W(h+x) = \frac{1}{2}\sigma A x$$

Now $x = \frac{\sigma l}{E}$

$$W\left(h + \frac{\sigma l}{E}\right) = \frac{1}{2}\cdot\sigma\cdot A\cdot\frac{\sigma l}{E}$$

or $$\sigma^2\left(\frac{Al}{2E}\right) - \sigma\left(\frac{Wl}{E}\right) - Wh = 0$$

$$\therefore \quad \sigma = \frac{\frac{Wl}{E} \pm \sqrt{\left(\frac{Wl}{E}\right)^2 + 4\left(\frac{Al}{2E}\right)Wh}}{Al/E}$$

$$= \frac{\frac{Wl}{E} \pm \sqrt{\left(\frac{Wl}{E}\right)^2 + 2\left(\frac{Al}{E}\right)Wh}}{Al/E}$$

$$= \frac{W}{A} \pm \sqrt{\left(\frac{W}{A}\right)^2 + \frac{2E}{AL}\times Wh}$$

$$= \frac{W}{A}\left[1 \pm \sqrt{1 + \frac{2AEh}{Wl}}\right]$$

Max. stress, $\sigma_{max} = \frac{W}{A}\left[1 + \sqrt{1 + \frac{2AEh}{Wl}}\right]$

IMP. CONCLUSIONS :- (i) If x is very small in comparison with h.

The work done by load $= W.h$

Equating the work done by the load to the strain energy stored in the rod

$$W.h = \frac{\sigma^2}{2E}.AL$$

$$\Rightarrow \quad \sigma^2 = \frac{2E\,W.h}{AL}$$

$$\therefore \quad \sigma = \sqrt{\frac{2EWh}{AL}}$$

(ii) If $h = 0$

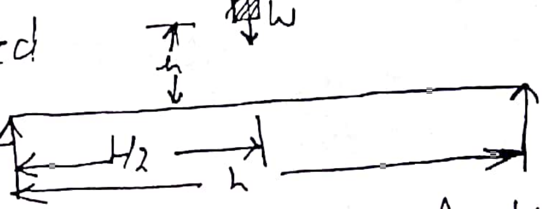
$$\therefore \quad \sigma = \frac{W}{A}\left[1 \pm \sqrt{1+0}\right]$$

$$\sigma_{max} = \frac{2W}{A}$$

which is the case of suddenly applied load.

Strain Energy due to impact on a simply supported beam

Let a weight W fall on a simply supported beam of length L at a mid-span through a height h



Let δ be the deflection produced in the beam. Then the work done by W is W(h+δ), provided the resultant stress produced in the beam within elastic limit.

If W_1 is the equivalent static load at mid-span to produce the deflection δ, then the work done by W_1 is $\frac{1}{2}W_1\delta$. The strain energy produced in the beam is the same in each case.

$$\frac{1}{2}W_1\delta = W(h+\delta)$$

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Now, $\delta = \frac{1}{48}\frac{W_1 L^3}{EI}$

$$\therefore \frac{W_1^3 L^3}{98EI} = W\left(h + \frac{W_1 L^3}{48EI}\right)$$

$$W_1^2 \frac{L^3}{96EI} - W_1 \frac{W L^3}{48EI} - Wh = 0$$

$$W_1^2 - 2WW_1 - \frac{98 WhEI}{L^3} = 0$$

$$\therefore \boxed{W_1 = W + \sqrt{W^2 + \frac{98WhEI}{L^3}}}$$

If $h = 0$, then $W_1 = 2W$

By knowing W_1, the max. bending moment is,

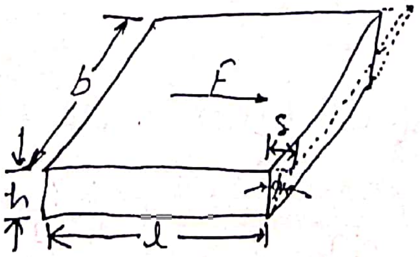
$$M = \frac{W_1 L}{4}$$

$$\therefore \boxed{\text{Bending stress, } \sigma = \frac{My}{I}}$$

Strain Energy due to shear stress :-

Consider a rectangular block of length l, width b & height h, rigidly fixed at base & subjected to a shearing force F.

Shearing stress $\tau = \frac{F}{bl}$



Deflection in the direction of F

$$\delta = \phi h$$

or $$\delta = \frac{\tau}{G} \cdot h$$

Shear strain energy $U = \frac{1}{2} F S$

$$= \frac{1}{2} \times \tau b l \cdot \frac{\tau}{G} h$$

$$= \frac{1}{2} \frac{\tau^2}{G} b l h$$

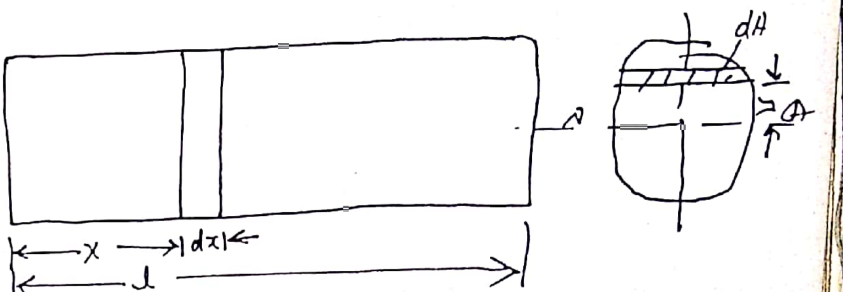
$$= \frac{\tau^2}{2G} \times \text{Vol.}$$

& Shear resilience $= \dfrac{\text{Shear strain energy}}{\text{Vol.}}$

$$= \frac{\tau^2}{2G}$$

Strain energy due to bending :-

Consider a small length dx of the beam at a distance x from left end. Let σ be the bending stress on the elementary area dA of the beam at distance y from the neutral axis.



Then $\sigma = \frac{My}{I}$

The strain energy stored in the [illegible] the bending stress is,

$$dU = \int \frac{\sigma^2}{2E} \times \text{vol.}$$

$$= \int \left(\frac{My}{I}\right)^2 \frac{1}{2E} \times dx . dA$$

$$= \frac{M^2}{2EI^2} dx \int y^2 dA$$

$$= \frac{M^2}{2EI^2} dx \; I \quad \left\{ \because I = \int y^2 dA \right\}$$

$$= \frac{M^2}{2EI} dx$$

∴ The flexural strain energy stored in the entire beam is

$$U = \int_0^l \frac{M^2 dx}{2EI} = \frac{M^2 l}{2EI} = \frac{1}{2} M\phi \,, \left\{ \text{where } \phi = \frac{Ml}{EI} = \frac{l}{R} \right\}$$

$$= \left(\frac{\sigma I}{y}\right)^2 \frac{l}{2EI} \qquad \left\{ \because M = \frac{\sigma I}{y} \right\}$$

$$= \frac{\sigma^2 I^2}{y^2} \frac{l}{2EI}$$

$$= \frac{\sigma^2 I}{y^2} \frac{l}{2E}$$

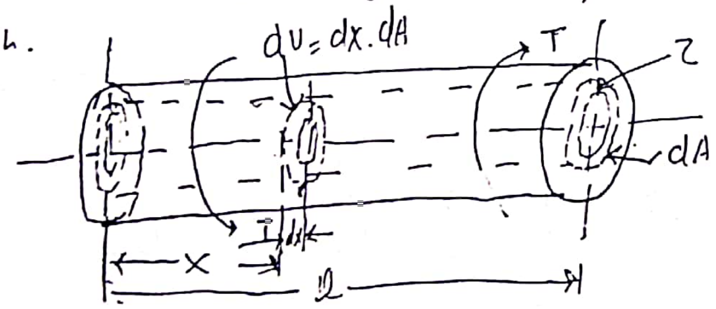
$$= \frac{\sigma^2}{2E} \times \frac{bd^3/12}{(d/2)^2} \cdot l \qquad \left\{ \because I = \frac{bd^3}{12} \;\&\; y = d/2 \right\}$$

$$= \frac{\sigma^2}{2E} \times \frac{bdl}{3}$$

$$= \frac{\sigma^2}{6E} \times \text{Vol.}$$

Strain energy due to torsion :-

Consider a shaft of length l subjected to a torque T as shown.



Consider a section of the shaft at a distance x from left end. Let τ be the shear stress acting on the elementary area dA at a distance r from the centre. Then the strain energy stored in the elementary strip due to torsion, known as the torsional strain energy is given by

$$dU = \int \frac{\tau^2}{2G} \times Vol.$$

$$= \int \left(\frac{1+\nu}{E}\right)\left(\frac{Tr}{J}\right)^2 \times Vol. \qquad \left\{\because \frac{T}{J} = \frac{\tau}{r}\right\}$$

$$= \left(\frac{1+\nu}{E}\right) \int \left(\frac{Tr}{J}\right)^2 dx\, dA$$

$$= \left(\frac{1+\nu}{E}\right) \frac{T^2}{J^2} dx \int r^2 dA$$

$$= \left(\frac{1+\nu}{E}\right) \frac{T^2 dx}{J^2} \cdot J = \left(\frac{1+\nu}{E}\right) \frac{T^2 dx}{J}$$

$\therefore$ Total strain energy stored in the shaft of uniform cross-section

$$U = \left(\frac{1+\nu}{E}\right) \int_0^l \frac{T^2 dx}{J}$$

$$= \left(\frac{1+\nu}{E}\right) \frac{T^2 l}{J}$$

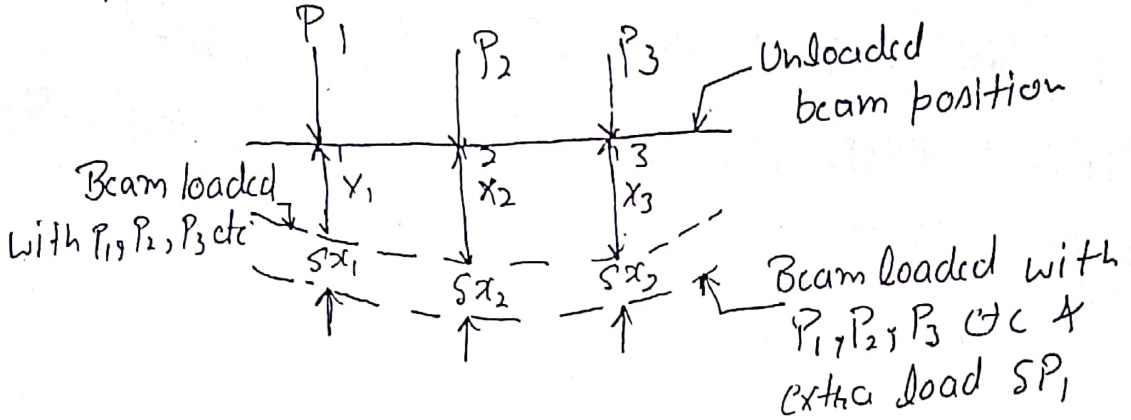
$$= \frac{T^2 l}{2GJ}$$

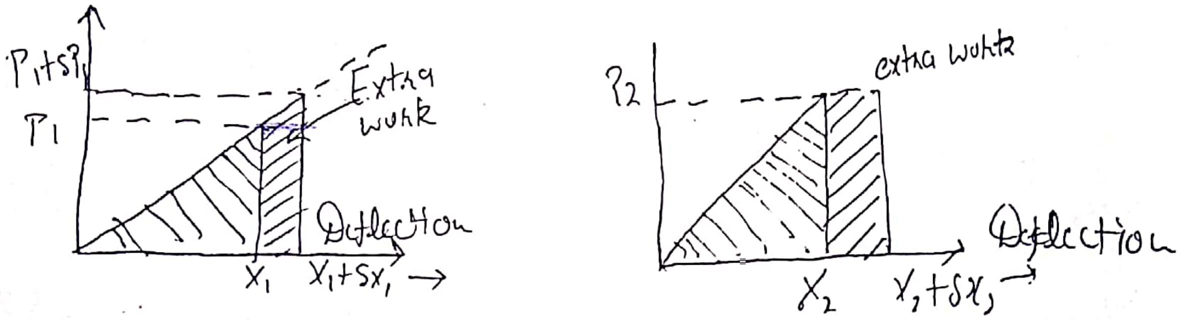
Now $\frac{T}{J} = \frac{G\theta}{l}$, $\theta = \frac{Tl}{JG}$

$$\boxed{\therefore U = \frac{1}{2} T\theta}$$

Castigliano's Theorem :-

This theorem states that the rate of change of strain energy with respect to statically independent force gives the component deflection of this force in the direction of the force.





(a)

In order to prove the theorem, consider the beam with forces P_1, P_2, P_3 etc. acting at points 1, 2, 3 etc. If x_1, x_2, x_3 etc. are shown in fig.

$$\therefore \quad U = \frac{1}{2}P_1x_1 + \frac{1}{2}P_2x_2 + \frac{1}{2}P_3x_3 + \dots \qquad (1)$$

Extra work done at 1, if P_1 is increased by δP_1 & deflection will be $\delta x_1, \delta x_2, \delta x_3$

$$= P_1 \times \delta x_1 + \cancel{\frac{1}{2}\delta P_1 \times \delta x_1}$$

Extra work done at 2,

$$= P_2 \times \delta x_2$$

Similarly extra work done at 3 etc. is $P_3\,\delta x_3$ etc.

Increase in strain energy = Total extra work done

$$\delta U = P_1 \delta x_1 + \frac{1}{2} \delta P_1 \delta x_1 + P_2 \delta x_2 + P_3 \delta x_3 + \ldots$$

Neglecting the product of small quantities, we get

$$\delta U = P_1 \delta x_1 + P_2 \delta x_2 + P_3 \delta x_3 + \ldots \quad (2)$$

If the loads $P_1 + \delta P_1$, P_2, P_3, etc were applied gradually from zero the total strain energy would be,

$$U + \delta U = \Sigma \frac{1}{2} \times \text{load} \times \text{Deflection}$$

$$= \frac{1}{2}(P_1 + \delta P_1)(x_1 + \delta x_1) + \frac{1}{2} P_2 (x_1 + \delta x_2)$$

$$+ \frac{1}{2} P_3 (x_3 + \delta x_3) + \ldots$$

$$= \frac{1}{2} P_1 x_1 + \frac{1}{2} P_1 \delta x_1 + \frac{1}{2} \delta P_1 x_1 + \frac{1}{2} \delta P_1 \partial x_1$$

$$+ \frac{1}{2} P_2 x_2 + \frac{1}{2} P_2 \delta x_2 + \frac{1}{2} P_3 x_3 + \frac{1}{2} P_3 \delta x_3 + \ldots \quad (3)$$

Neglecting the product of small quantities & subtracting eqn. (1) from (3)

$$\delta U = \frac{1}{2} P_1 \delta x_1 + \frac{1}{2} \delta P_1 x_1 + \frac{1}{2} P_2 \delta x_2 + \frac{1}{2} P_3 \delta x_3 + \ldots$$

$$2\delta U = \delta P_1 x_1 + P_1 \delta x + P_2 \delta x_2 + P_3 \delta x_3 + \ldots \quad (4)$$

Subtracting eqn (2) from (4), we get

$$\delta U = \delta P_1 \times x_1$$

$$\frac{\delta U}{\delta P_1} = x_1$$

Or In the limit $\frac{\partial U}{\partial P_2} = x_2$

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Max Well's Theorem:-

Maxwell's Theorem :-

Consider an elastic body constrained in space & acted upon by point force P_i. $i = 1, 2, \ldots n$. Then the strain energy from this system of forces is given by

$$U_I = \sum_{i=1}^{n} \frac{1}{2} P_i S_i$$

where S_i = Corresponding deflections under the loads P_i.

Now consider another loading process for the body where in a new set of point forces $P_j = 1, 2, \ldots, m$ are applied. Then the strain energy from this system of force is given by:

$$U_{II} = \sum_{j=1}^{m} \frac{1}{2} P_j S_j$$

Let us now, study the strain energy of the body under the combined action of both forces system I & II.

We therefore apply loading system I producing strain energy U_I is given by:

$$U_I = \frac{1}{2} \sum_{i=1}^{n} (P_i)_I (S_i)_I$$

Now apply load system II to the body. Since force system I remains constant during the application of force system II, the strain energy $U_{I,II}$ becomes:

$$U_{I,II} = \sum_{i=1}^{n} (P_i)_I (S_i)_{II}$$

where $(S_i)_{II}$ = the displacement due to force system II at the position of the point of application of force system II at the position of the point of application of force $(P_i)_I$ of system I.

The increase in strain energy U_{II} is

$$U_{II} = \frac{1}{2} \sum_{j=1}^{m} (P_j)_{II} (\delta_j)_{II}$$

The total strain energy becomes:

$$U = U_I + U_{I,II} + U_{II}$$

Now let us reload the body in the reverse order. The strain energy U' becomes:

$$U' = U_{II} + U_{II,I} + U_I$$

where $U_{II,I} = \sum_{j=1}^{m} (P_j)_{II} (\delta_j)_I$

However, according to the principle of superposition,

$$U = U'$$

$$\therefore U_{I,II} = U_{II,I}$$

or $$\sum_{i=1}^{n} (P_i)_I (\delta_i)_{II} = \sum_{j=1}^{m} (P_j)_{II} (\delta_j)_I$$

This is Maxwell reciprocal theorem. It states that the workdone by the first system of loads due to displacements caused by a second system of loads equals the work done by the second system of loads due to displacements caused by first system of loads.