Theories of Elastic Failure

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A material is said to be failed when it is stressed beyond the elastic limit & permanent deformation occurs. Failure of a material represents either direct separation of particles from each other (brittle fracture) or slipping of particles (ductile fracture or yielding) accompained by considerable plastic deformations.

If the material is subjected to a single type of stress (i.e. axial, bending or torsional), then it is very easy to anticipate when the failure is likely to occur. If, however the material is subjected to a complex stress system, then it is not easy to predict the failure of material straight away. In a complex stress system the quantities like tensile stress, shear stress & strain energy, ~~also~~ can be calculated from the known stresses & material constants, & the problem is to decide which quantity is the criterion of failure, i.e. the cause of the material passing beyond its elastic limit & taking up a permanent set. Having decided, the actual value of that particular factor which corresponds to the onset of failure is usually taken to be the value it reaches in the simple tension case at the elastic limit.

In order to do this, the following theories have been put forward:

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(i) Maximum Principal Stress Theory or Rankine Theory.

(ii) Maximum Principal Strain Theory or St. Venant's Theory

(iii) Maximum Shear Stress Theory or Stress Difference Theory or Guest - Coulomb's Theory or Tresca's Theory

(iv) Maximum Strain Energy Theory or Beltrami - Haigh's Theory.

(v) Maximum Shear Strain Energy or Distortion energy theory or Mises - Hency Theory

(vi) Mohr's Theory.

(1) Maximum Principal Stress Theory :- According to this theory failure of the material is assumed to have taken place under a state of complex stresses when the value of maximum principal stress σ_1 reaches, a value equal to that of elastic limit stress σ as found in a simple tensile test. i.e.

$$\sigma_1 = \sigma$$

This theory has been found to predict the failure behaviour of brittle materials, like cast iron satisfactorily. The elastic limit stress in tension & compression is assumed to be the same.

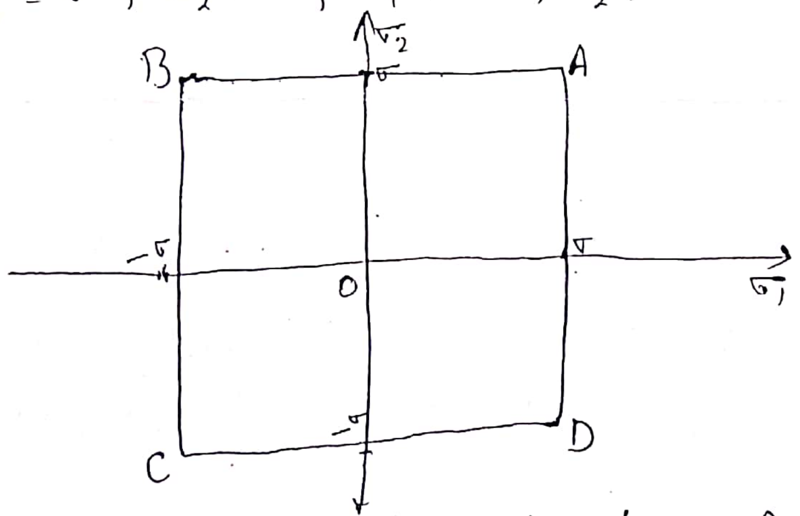
Note:- Let $\sigma_1, \sigma_2, \sigma_3$ be the principal stresses at a point in a body taken in descending order & σ is the

Magnitude of direct stress in simple tension at the elastic limit (for all theories)

Graphical Representation :- Where only a two-dimensional stress system is under consideration the limits of principal stress can be shown graphically according to different theories.

Maximum principal stress equal numerically to the elastic limit. This produces a square boundary ABCD, the sides being defined by

$$\sigma_1 = \sigma,\quad \sigma_2 = \sigma,\quad \sigma_1 = -\sigma,\quad \sigma_2 = -\sigma$$



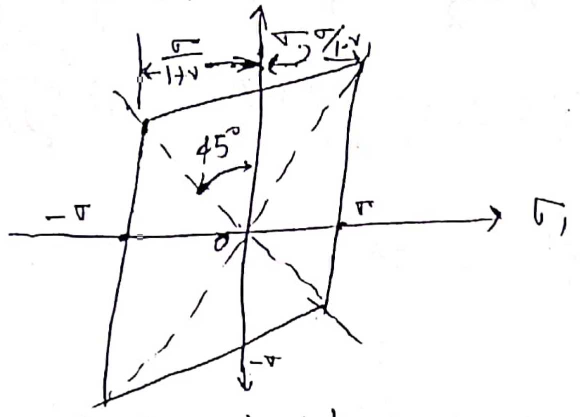
Any point having coordinates (σ_1, σ_2) falling outside the square will indicate failure.

(ii) Maximum Principal strain theory : According to this theory, failure of material is deemed to have taken place when the maximum principal strain reaches a value calculated from a simple tensile test.

Thus $$\frac{1}{E}\left[\sigma_1 - \nu(\sigma_2 + \sigma_3)\right] = \frac{\sigma}{E}$$

This theory over estimates the behaviour of ductile materials.

Graphical Representation :- For yielding in tension this theory states that $\sigma_1 - \nu\sigma_2 = \sigma$ & for compressive yield, with σ_2 compressive $\sigma_2 - \nu\sigma_1 = \sigma$



Rhomboid

These equations when plotted produce the rhomboid failure envelope.

(iii) Maximum Shear Stress Theory :- According to this theory the failure of the material is deemed to have taken place when the maximum shear stress exceeds the maximum shear stress in a simple tensile test, i.e.

$$\frac{\sigma_1 - \sigma_3}{2} = \sigma/2$$

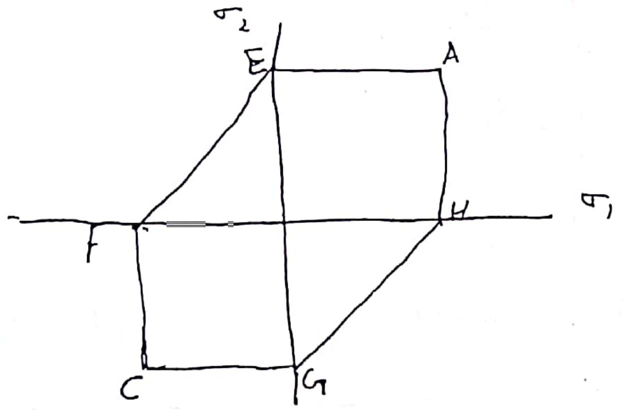
$$\therefore \quad \sigma_1 - \sigma_3 = \sigma$$

This theory has been found to give quite satisfactory results for ductile materials.

Graphical Representation :- Max. shear stress equal numerically to the value in simple tension ($\pm\sigma$). Where the principal stresses are alike, the greatest maximum shear stress is $\frac{1}{2}\sigma_1$ (or $\frac{1}{2}\sigma_2$), obtained by taking half the difference between the principal stresses σ_1 & 0, or σ_2 & 0. This produces line

$$\frac{1}{2}\sigma_1 = \frac{1}{2}\sigma, \quad \frac{1}{2}\sigma_2 = \frac{1}{2}\sigma, \quad \frac{1}{2}\sigma_1 = -\frac{1}{2}\sigma, \quad \& \ \frac{1}{2}\sigma_2 = -\frac{1}{2}\sigma$$

in the first & third quadrants (HA, AE, FC, CG).



When the principal stresses are of opposite type, Max. shear stress is

$$\frac{1}{2}(\sigma_1 - \sigma_2) = \pm \frac{1}{2}\sigma$$

completing the figure in the second & forth quadrants with the lines EF & GH. The boundary is then AEFCGHA.

) Maximum Strain Energy Theory :- According to this theory failure is assumed to take place when the total strain energy exceeds the strain energy determined from a simple tensile test. Thus

$$\frac{1}{2E}\left[\sigma_1^2+\sigma_2^2+\sigma_3^2-2\nu(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)\right] = \frac{\sigma^2}{2E}$$

Or $\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) = \sigma^2$

This theory has been found to give good results for ductile materials.

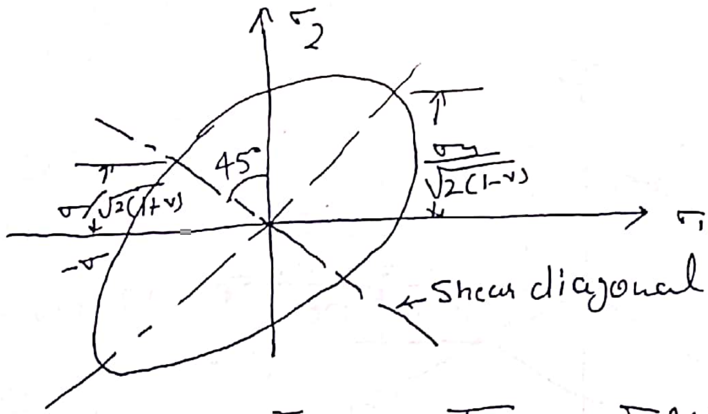
Graphical Representation :-

When $\sigma_3 = 0$, this theory becomes

$$\sigma_1^2 + \sigma_2^2 - 2\nu\sigma_1\sigma_2 = \sigma^2$$

or $\left(\frac{\sigma_1}{\sigma}\right)^2 + \left(\frac{\sigma_2}{\sigma}\right)^2 - 2\nu\left(\frac{\sigma_1}{\sigma}\right)\left(\frac{\sigma_2}{\sigma}\right) = 1$

This is the equation of an ellipse with semi-major & semi-minor axes $\frac{\sigma}{\sqrt{1-\nu}}$, $\frac{\sigma}{\sqrt{1+\nu}}$ respectively, each at 45° to the coordinate axes.



Max. Strain Energy Theory Ellipse.

(V) Maximum Shear Strain Energy Theory :-

According to this theory failure is assumed to take place when the maximum shear strain energy exceeds the shear strain energy in a simple tensile test, i.e.

$$\frac{1}{12G}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]=\frac{\sigma^2}{6G}$$

Or $$(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2=2\sigma^2$$

This theory gives very satisfactory results for the ductile materials.

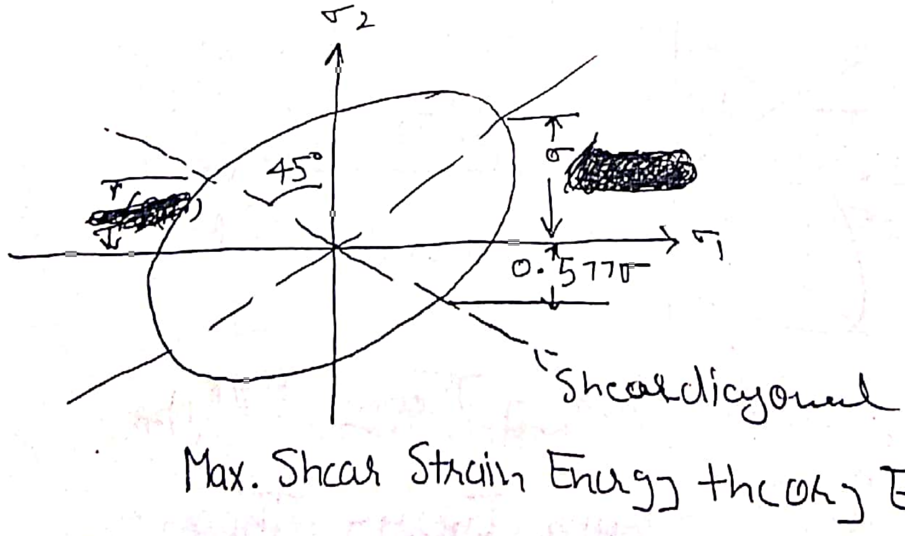
(vii) Graphical Representation :-

When $\sigma_3 = 0$, this theory becomes

$$\sigma_1^2+\sigma_1^2-\sigma_1\sigma_2=\sigma^2$$

Or $$\left(\frac{\sigma_1}{\sigma}\right)^2+\left(\frac{\sigma_2}{\sigma}\right)^2-\left(\frac{\sigma_1}{\sigma}\right)\left(\frac{\sigma_2}{\sigma}\right)=0$$

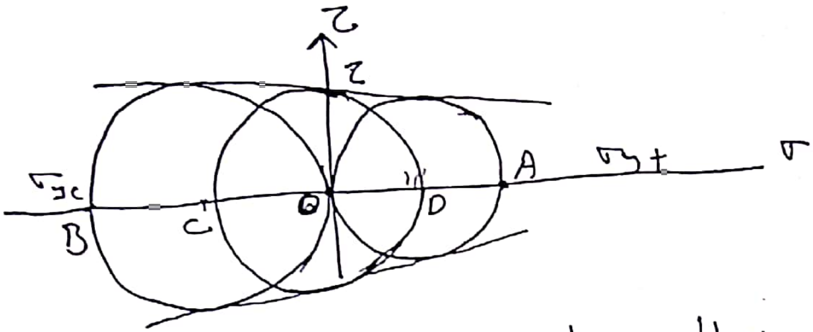
This is the equation of an ellipse with semi-major & semi-minor axes $\sqrt{2}\sigma$ & $\sqrt{2/3}\,\sigma$ respectively at 45° to the co-ordinate axes



Max. Shear Strain Energy theory Ellipse.

(vi) Mohr's Theory :- Mohr has proposed a theory to predict the failure of brittle material on his stress circle.

The circle on diameter OA is that of pure tension, the circle on diameter OB that for pure compression & the circle with centre O & diameter CD is that for pure shear.



An envelope to these circles then represents the failure envelope according to Mohr theory.
A failure condition is then indicated when a stress circle for a particular complex stress condition is found to cut the envelope.
An expression for Mohr's theory may be derived as follows with principal stresses σ_1 & σ_2, which is

$$\frac{\sigma_1}{\sigma_{yt}} + \frac{\sigma_2}{\sigma_{yc}} = 1$$

& known as modified Mohr's shear stress theory for brittle materials.