THICK PRESSURE VESSELS

All the shells subjected to fluid pressure may be called pressure vessels. Pressure vessels may be defined as thin or thick depending upon the ratio of wall thickness to the radius of the shell.

Generally, when the ratio of wall thickness t of the shell to mean radius R is less than $1/15$, then the shell may be termed as thin shell. $t/R < 1/15$

When the ratio of wall thickness t of the shell to mean radius R is greater than $1/15$, then the shell may be termed as thick vessel.

LAME'S EQUATION :-

Consider a thick cylinder of length l, internal radius r_1, & external radius r_2, subjected to internal & external uniformly distributed pressures of intensities p_1 & p_2 respectively. The following assumptions are made.

Let σ_r = radial stress at any radius r

σ_θ = Hoop stress at same radius r

σ_z = Uniform Longitudinal stress

$$\therefore \varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_\theta + \sigma_r)\right]$$

Since $\varepsilon_z, E, \sigma_z$ & ν are all constant, (so) $\sigma_\theta + \sigma_r = 2A$ (1)

For the cylinder, consider an annular ring of cylinder between radii r & $r+\delta r$. Let the internal radial stress in this ring be σ_r & external radial stress be $\sigma_r + \delta\sigma_r$.

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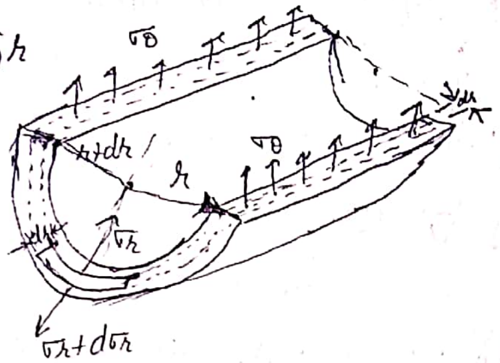
Considering the equillibrium of half the ring, we find first:

Bursting force $= (\sigma_r + \delta\sigma_r)\,2(r+\delta r).l - \sigma_r.2r.l$

Simplifying & neglecting small quantities, we get

$= 2\sigma_r.\delta r\,l + 2\delta\sigma_r.rl$

Resisting force $= 2\sigma_\theta.l\,\delta r$



For equillibrium of ring,

$2\sigma_\theta.l\,\delta r = 2\sigma_r\,\delta r\,l + 2\delta\sigma_r\,rl$

$$\sigma_\theta = \sigma_r + r\frac{\delta\sigma_r}{\delta r}$$

In the limiting case

$$\sigma_\theta = \sigma_r + r\frac{d\sigma_r}{dr} \quad -(2)$$

$$= \frac{d(r\sigma_r)}{dr}$$

From eqn. (1) & (2)

$$\frac{d(r\sigma_r)}{dr} + \sigma_r = 2A$$

or $$r\frac{d\sigma_r}{dr} = 2(A - \sigma_r)$$

$$\frac{d\sigma_r}{dr} = \frac{2(A-\sigma_r)}{r}$$

or $$\frac{d\sigma_r}{A-\sigma_r} = \frac{2dr}{r}$$

Integrating, we get

$$\int \frac{d\sigma_r}{A-\sigma_r} = 2\int\frac{dr}{r}$$

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$\Rightarrow -\log(A-\sigma_r) = 2\log r - \log B$

$\Rightarrow \log(A-\sigma_r) = -2\log r + \log B$

$\Rightarrow \log(A-\sigma_r) = \log\frac{B}{r^2}$

$\therefore A-\sigma_r = \frac{B}{r^2}$

$$\therefore \sigma_r = A - \frac{B}{r^2} \quad -(3)$$

where B is another constant.

$\therefore \sigma_\theta + A - \frac{B}{r^2} = 2A$ {∵ From eqn (1)}

$$\sigma_\theta = A + \frac{B}{r^2} \quad -(4)$$

The equation (3) & (4) are known as Lame's equations.

Case I :- (i) At $r = r_1$, $\sigma_r = -p_1$

(ii) At $r = r_2$, $\sigma_r = -p_2$

$\therefore -p_1 = A - \frac{B}{r_1^2}$

$-p_2 = A - \frac{B}{r_2^2}$

$\Rightarrow p_1 - p_2 = B\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right)$

$\Rightarrow B = \frac{r_1^2 r_2^2 (p_1 - p_2)}{r_2^2 - r_1^2}$

$A = \frac{B}{r_1^2} - p_1 = \frac{r_2^2(p_1-p_2)}{r_2^2 - r_1^2} - p_1$

$= \frac{\cancel{r_2^2 p_1} - r_2^2 p_2 - \cancel{r_2^2 p_1} + r_1^2 p_1}{r_2^2 - r_1^2}$

$= \frac{p_1 r_1^2 - p_2 r_2^2}{r_2^2 - r_1^2}$

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$$\therefore \boxed{\sigma_r = \frac{p_1 r_1^2 - p_2 r_2^2}{r_2^2 - r_1^2} - \frac{r_1^2 r_2^2 (p_1 - p_2)}{r^2 (r_2^2 - r_1^2)}} \quad -(5)$$

$$\boxed{\sigma_\theta = \frac{p_1 r_1^2 - p_2 r_2^2}{r_2^2 - r_1^2} + \frac{r_1^2 r_2^2 (p_1 - p_2)}{r^2 (r_2^2 - r_1^2)}} \quad -(6)$$

Case II : (a) Internal pressure only i.e. $p_2 = 0$ then

$$\therefore \sigma_r = \frac{p_1 r_1^2}{r_2^2 - r_1^2} - \frac{r_1^2 r_2^2 p_1}{r^2 (r_2^2 - r_1^2)}$$

$$\boxed{\sigma_r = \frac{p_1 r_1^2}{r_2^2 - r_1^2}\left[1 - \frac{r_2^2}{r^2}\right]} \quad -(7)$$

$$\& \quad \sigma_\theta = \frac{p_1 r_1^2}{r_2^2 - r_1^2} + \frac{r_1^2 r_2^2 p_1}{r^2 (r_2^2 - r_1^2)}$$

$$\boxed{\sigma_\theta = \frac{p_1 r_1^2}{r_2^2 - r_1^2}\left[1 + \frac{r_2^2}{r^2}\right]} \quad -(8)$$

(b) External pressure only i.e. $p_1 = 0$ then

$$\sigma_r = \frac{-p_2 r_2^2}{r_2^2 - r_1^2} + \frac{r_1^2 r_2^2 p_2}{r^2 (r_2^2 - r_1^2)}$$

$$\boxed{\sigma_r = \frac{p_2 r_2^2}{r_2^2 - r_1^2}\left[\frac{r_1^2}{r^2} - 1\right]} \quad -(9)$$

$$\boxed{\sigma_\theta = \frac{-p_2 r_2^2}{r_2^2 - r_1^2}\left[1 + \frac{r_1^2}{r^2}\right]} \quad -(10)$$

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(3)

(c) Solid cylinder having external radial pr.

i.e.

$r_1 = 0$, $p_1 = 0$

$\sigma_r = -p_2$

$\sigma_\theta = -p_2$

$$\therefore \boxed{\sigma_r = \sigma_\theta = -p_2}$$

(d) Longitudinal stress:-

Consider now the cross-section of a thick cylinder with closed ends subjected to an internal pressure p_1 & external pressure p_2. For horizontal equilibrium

$$p_1 \pi r_1^2 - p_2 \times \pi r_2^2 = \sigma_z \times \pi (r_2^2 - r_1^2)$$

$$\sigma_z = \frac{p_1 r_1^2 - p_2 r_2^2}{r_2^2 - r_1^2}$$

(e) Maximum shear stress:-

$$\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_2)$$

$$= \frac{1}{2}(\sigma_\theta - \sigma_r)$$

$$= \frac{1}{2}\left(A + \frac{B}{r^2} - A + \frac{B}{r^2}\right) = \frac{B}{r^2}$$

Shear stress will be max at $r = r_1$

$$\boxed{\tau_{max} = \frac{r_2^2 (p_1 - p_2)}{r_2^2 - r_1^2}}$$

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Radial Deflection :- Let u be the increase in radius r.

$\therefore$ Hoop strain $$\varepsilon_\theta = \frac{2\pi(r+u) - 2\pi r}{2\pi r}$$

$$= \frac{2\pi r + 2\pi u - 2\pi r}{2\pi r}$$

$$= \frac{u}{r} \quad -(1)$$

& Also, Hoop strain, $$\varepsilon_\theta = \frac{1}{E}[\sigma_\theta - \nu\sigma_r] \quad -(2)$$

from (1) & (2)

$$\therefore \frac{u}{r} = \frac{1}{E}[\sigma_\theta - \nu\sigma_r]$$

$\therefore$ Radial Deflection $$u = \frac{r}{E}[\sigma_\theta - \nu\sigma_r]$$

For the cylinder subjected to internal pr. only.

$$\delta_1 = \frac{r}{E}\left[\frac{p_1 r_1^2}{r_2^2 - r_1^2}\left\{1 + \frac{r_2^2}{r^2}\right\} - \nu\left(\frac{p_1 r_1^2}{r_2^2 - r_1^2}\right)\left\{1 - \frac{r_2^2}{r^2}\right\}\right]$$

$$= \frac{r}{E}\left[(1-\nu) + (1+\nu)\frac{r_2^2}{r^2}\right] \times \frac{p_1 r_1^2}{r_2^2 - r_1^2}$$

At $r = r_1$

$$\boxed{\delta_1 = \frac{p_1 r_1}{E}\left[\frac{r_1^2 + r_2^2}{r_2^2 - r_1^2} + \nu\right]}$$

For the cylinder subjected to external pr. only,

$$\delta_2 = \frac{r}{E}\left[\left\{1 + \frac{r_1^2}{r^2}\right\} + \nu\left\{\frac{r_1^2}{r^2} - 1\right\}\right] \times \left(\frac{-p_2 r_2^2}{r_2^2 - r_1^2}\right)$$

$$= \frac{r}{E}\left[(1-\nu) + (1+\nu)\frac{r_1^2}{r^2}\right] \times \frac{-p_2 r_2^2}{r_2^2 - r_1^2}$$

At $r = r_2$

$$\boxed{\delta_2 = \frac{-p_2 r_2}{E}\left[\frac{r_1^2 + r_2^2}{r_2^2 - r_1^2} - \nu\right]}$$

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Compound Cylinder :- In a thick cylinder subjected to internal pressure, the maximum hoop stress occurs on the inner surface & diminishes to a minimum value on the outer surface. The intensity of the internal pressure which a cylinder of given thickness can carry in, therefore, limited by maximum hoop stress which should not exceed the permissible tensile stress for the material. In order to reduce stresses inside a thick cylinder subjected to high internal pressure, two or more cylinders are assembled together by 'shrink fitting'. In case of two cylinders, the outside cylinder has internal diameter slightly less than the external diameter of the inside cylinder. The outside cylinder is expanded by heating & the two cylinders are assembled together. After the cylinders attain the same temperature, the outside cylinder causes circumferential compressive stresses in the inner cylinder & the inner cylinders causes circumferential tensile stresses in the outside cylinder. When this compound cylinder is subjected to internal pressure, tensile hoop stresses shall be set up throughout the material of the cylinder. The final stresses will be algebraic sum of the initial shrinkage stresses & the stresses due to internal pressure.

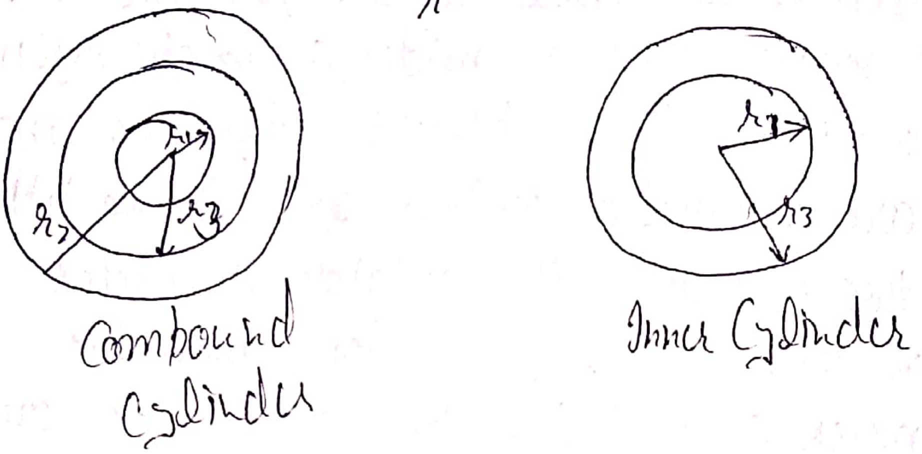
(a) Determination of Resultant Stresses :- Consider a compound cylinder of inner radius r_1, outer radius r_2 & junction radius r_3

Let p_s = Shrinkage pr. at the common radius r_3

(i) <u>Shrinkage stresses</u>: For the inner cylinder

$$\sigma_r = A - \frac{B}{r^2}$$

$$\sigma_\theta = A + \frac{B}{r^2}$$



At $r = r_1$ $\sigma_r = 0$

& at $r = r_3$, $\sigma_r = -p_s$

$$\therefore \quad 0 = A - \frac{B}{r_1^2}$$

$$-p_s = A - \frac{B}{r_3^2}$$

Hence $A = \frac{B}{r_1^2}$

$$\therefore \quad -p_s = B\left(\frac{1}{r_1^2} - \frac{1}{r_3^2}\right)$$

$$\Rightarrow B = -\frac{r_1^2 r_3^2 p_s}{(r_3^2 - r_1^2)}$$

$$A = -\frac{r_3^2 p_s}{r_3^2 - r_1^2}$$

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$$\therefore \sigma_r = -\frac{r_3^2 p_r}{r_3^2 - r_1^2} + \frac{r_1^2 r_3^2 p_r}{r^2 (r_3^2 - r_1^2)}$$

$$= \frac{r_3^2 p_r}{(r_3^2 - r_1^2)} \left[-1 + \frac{r_1^2}{r^2} \right]$$

$$\sigma_\theta = \frac{-r_3^2 p_r}{r_3^2 - r_1^2} \left[1 + \frac{r_1^2}{r^2} \right]$$

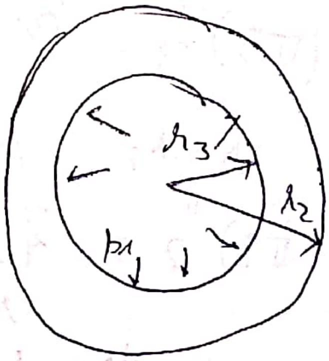
At $r = r_3$

$$(\sigma_\theta)'_3 = -p_r \left(\frac{r_3^2 + r_1^2}{r_3^2 - r_1^2} \right)$$

At $r = r_1$

$$(\sigma_\theta)'_1 = \frac{-2 r_3^2 p_r}{r_3^2 - r_1^2}$$

<u>For outer Cylinder</u> :-



$$\sigma_r = a - \frac{b}{r^2}$$

$$\sigma_\theta = a + \frac{b}{r^2}$$

At $r = r_3$, $\sigma_r = -p_r$

$r = r_2$ $\quad \sigma_r = 0$

$$\therefore -p_r = a - \frac{b}{r_3^2}$$

$$0 = a - \frac{b}{r_2^2}$$

$$\therefore a = \frac{b}{r_2^2}$$

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$$-p_1 = \frac{b}{r_2^2} - \frac{b}{r_3^2}$$

$$\therefore \quad b = \frac{p_1 r_2^2 r_3^2}{(r_2^2 - r_3^2)}$$

$$\& \quad a = \frac{p_1 r_3^2}{r_2^2 - r_3^2}$$

$$\therefore \quad \sigma_r = \frac{p_1 r_3^2}{r_2^2 - r_3^2} - \frac{p_1 r_2^2 r_3^2}{r^2(r_2^2 - r_3^2)}$$

$$= \frac{p_1 r_3^2}{r_2^2 - r_3^2}\left[1 - \frac{r_2^2}{r^2}\right]$$

$$\& \quad \sigma_\theta = \frac{p_1 r_3^2}{r_2^2 - r_3^2}\left[1 + \frac{r_2^2}{r^2}\right]$$

At $r = r_3$

$$(\sigma_\theta)''_3 = p_1\left[\frac{r_3^2 + r_2^2}{r_2^2 - r_3^2}\right]$$

At $r = r_2$

$$(\sigma_\theta)''_2 = \frac{2 p_1 r_3^2}{r_2^2 - r_3^2}$$

(ii) Stresses due to internal fluid pressure

let p_i = Internal fluid pressure

Assuming that the compound cylinder behaves as one unit

$$\sigma_r = \alpha - \frac{B}{r^2}$$

$$\sigma_\theta = \alpha + \frac{B}{r^2}$$

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$\therefore$ At $r = r_1$ $\quad \sigma_r = -p_i$

$r = r_2$ $\quad \sigma_r = 0$

$\therefore -p_i = \alpha - \dfrac{\beta}{r_1^2}$

$0 = \alpha - \dfrac{\beta}{r_2^2}$

$\therefore \alpha = \dfrac{\beta}{r_2^2}$

$-p_i = \beta\left(\dfrac{1}{r_2^2} - \dfrac{1}{r_1^2}\right)$

$\therefore \beta = \dfrac{p_i r_1^2 r_2^2}{r_2^2 - r_1^2}$

& $\alpha = \dfrac{p_i r_1^2}{r_2^2 - r_1^2}$

$\sigma_r = \dfrac{p_i r_1^2}{r_2^2 - r_1^2}\left[1 - \dfrac{r_2^2}{r^2}\right]$

& $\sigma_\theta = \dfrac{p_i r_1^2}{r_2^2 - r_1^2}\left[1 + \dfrac{r_2^2}{r^2}\right]$

$\therefore$ At $r = r_1$

$(\sigma_\theta)_1 = p_1\left(\dfrac{r_2^2 + r_1^2}{r_2^2 - r_1^2}\right)$

At $r = r_3$

$(\sigma_\theta)_3 = p_1 \dfrac{r_1^2}{r_3^2}\left[\dfrac{r_3^2 + r_2^2}{r_2^2 - r_1^2}\right]$

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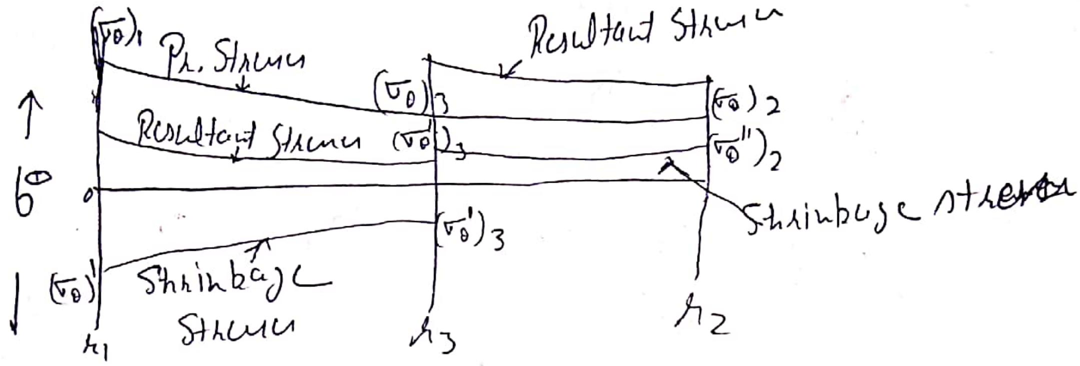
At $r = r_2$

$$(\sigma_\theta)_2 = \frac{2 p_i r_1^2}{r_2^2 - r_1^2}$$



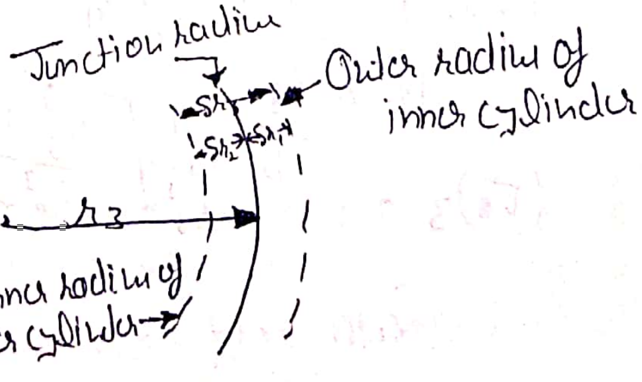
Distribution of stress in Compound Cylinder

Resultant Stress

Radius	Cylinder	Shrinkage Stress	Stress due to fluid pr.	Resultant Stress
r_1	Inner	$(\sigma_\theta)_1'$	$(\sigma_\theta)_1$	$(\sigma_\theta)_1 + (\sigma_\theta)_1'$
r_3	Inner	$(\sigma_\theta)_3'$	$(\sigma_\theta)_3$	$(\sigma_\theta)_3 + (\sigma_\theta)_3'$
r_3	Outer	$(\sigma_\theta)_3''$	$(\sigma_\theta)_3$	$(\sigma_\theta)_3 + (\sigma_\theta)_3''$
r_2	Outer	$(\sigma_\theta)_2''$	$(\sigma_\theta)_2$	$(\sigma_\theta)_2 + (\sigma_\theta)_2''$

(b) <u>Shrinkage Allowance</u>:-

The inner radius of the outer cylinder shall be slightly smaller than the outer radius of inner cylinder, so that when the cylinder is shrunk on, the required radical pressure p_s is achieved at the common surface.



Let r_3 = Junction radius after shrinking on

δr_1 = Difference b/w the outer radius of inner cylinder & r_3

δr_2 : Difference between the inner radius of outer cylinder & r_3

$\delta r_3 = \delta r_1 + \delta r_2$

= Shrinkage allowance,

i.e. the original difference of radii of cylinders at the junction.

(i) Cylinders made of same material

i.e $E = E_1 = E_2$, $\nu = \nu_1 = \nu_2$

For the inner cylinder, the circumferential strain at the common radius r_3 is given by:

$$\frac{\delta r_1}{r_3} = \frac{1}{E}\left[(\sigma_\theta)'_3 - \nu(\sigma_r)'_3\right]$$

where $(\sigma_\theta)'_3$ = hoop stress at the junction due to shrinking

$$= -p_s\left(\frac{r_3^2 + r_1^2}{r_3^2 - r_1^2}\right)$$

& $(\sigma_r)'_3 = -p_s$

$$\frac{\delta r_1}{r_3} = \frac{-p_s}{E}\left[\frac{r_3^2 + r_1^2}{r_3^2 - r_1^2} - \nu\right] \quad -(1)$$

Similarly for outer cylinder, the circumferential strain at the common radius r_3 is given by:

$$\frac{\delta r_2}{r_3} = \frac{1}{E}\left[(\sigma_\theta)''_3 - \nu(\sigma_r)''_3\right]$$

$$= \frac{1}{E}\left[p_s\left(\frac{r_3^2 + r_2^2}{r_2^2 - r_3^2}\right) - \nu(-p_s)\right]$$

$$= \frac{p_s}{E}\left[\frac{r_3^2 + r_2^2}{r_2^2 - r_3^2} + \nu\right]$$

By taking the algebraic sum, we get

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$$\frac{\delta r_1 + \delta r_2}{r_3} = \frac{pr}{E}\left[\left(\frac{r_3^2 + r_1^2}{r_3^2 - r_1^2}\right) + \left(\frac{r_3^2 + r_2^2}{r_2^2 - r_3^2}\right)\right]$$

$$\therefore \boxed{\frac{\delta r_3}{r_3} = \frac{pr}{E}\left[\frac{r_3^2 + r_1^2}{r_3^2 - r_1^2} + \frac{r_3^2 + r_2^2}{r_2^2 - r_3^2}\right]}$$

If $\Delta\theta$ = rise in temperature

α = co-efficient of linear exansion

Then $\boxed{\delta r_3 = r_3 \Delta\theta \cdot \alpha}$

(ii) Cylinders made of different materials

$$\frac{\delta r_1}{r_3} = \frac{pr}{E_1}\left[\frac{r_3^2 + r_1^2}{r_3^2 - r_1^2} - \nu_1\right]$$

& $$\frac{\delta r_2}{r_3} = \frac{pr}{E_2}\left[\frac{r_2^2 + r_3^2}{r_2^2 - r_3^2} + \nu_2\right]$$

Hence $$\frac{\delta r_3}{r_3} = \frac{\delta r_1 + \delta r_2}{r_3}$$

$$\boxed{\frac{\delta r_3}{r_3} = pr\left[\frac{1}{E}\left\{\frac{r_3^2 + r_1^2}{r_3^2 - r_1^2} - \nu_1\right\} + \frac{1}{E_2}\left\{\frac{r_2^2 + r_3^2}{r_2^2 - r_3^2} + \nu_2\right\}\right]}$$

& $$\boxed{\frac{\delta r_3}{r_3} = r_3 \Delta\theta \alpha}$$

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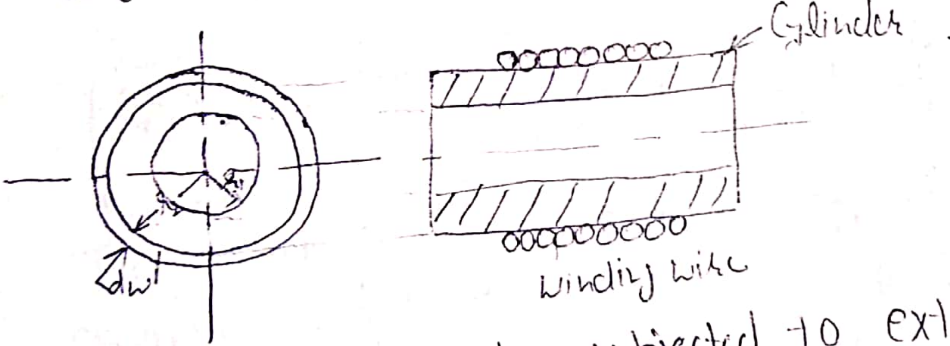
Wire Wound Cylinders :-

(8)

In order to strengthen the tube against the application of internal pressure it may first be wound with wire under tension, thus putting the wall in compression. When the pressure is applied the final hoop stress produced is much less than it would be without the wire reinforcement. The maximum stress will probably be that in the wire, which must be made of a high tensile material.

Analysis for stresses

(a) Winding stresses: Consider a thick cylinder of inner radius r_1, outer radius r_2, wound with a wire diameter of d_w with a constant tension T in the wire.



For a thick cylinder subjected to external pressure p_2 only, the stresses at a radius r are given by

$$\sigma_r = \frac{-p_2 r_2^2}{r_2^2 - r_1^2}\left[1 - \frac{r_1^2}{r^2}\right] \quad (a)$$

$$\sigma_\theta = \frac{-p_2 r_2^2}{r_2^2 - r_1^2}\left[1 + \frac{r_1^2}{r^2}\right] \quad (b)$$

Dividing equation (b) by (a), we get

$$\frac{\sigma_\theta}{\sigma_r} = \frac{r^2 + r_1^2}{r^2 - r_1^2} \quad (c)$$

Also considering the equilibrium of the annular ring of the cylinder between radii r & $r+dr$, we get

$$\sigma_\theta = \sigma_r + r\frac{d\sigma_r}{dr} \qquad (d),$$

When the cylinder is wound with a constant winding tension T in the wire, the value of σ_θ in the winding is reduced below T by the hoop compression given by eqn. (c). Hence final hoop stress in the winding at radius r may be written as:

$$\sigma_\theta = T + \sigma_r\left(\frac{r^2+r_1^2}{r^2-r_1^2}\right) \qquad (e)$$

By substituting eqn (d) in (e), we get

$$\sigma_r + r\frac{d\sigma_r}{dr} = T + \sigma_r\left(\frac{r^2+r_1^2}{r^2-r_1^2}\right)$$

$$T + \sigma_r\left[\frac{r^2+r_1^2-r^2+r_1^2}{r^2-r_1^2}\right] = r\frac{d\sigma_r}{dr}$$

Or $$T + \frac{2r_1^2}{(r^2-r_1^2)}\sigma_r = r\frac{d\sigma_r}{dr}$$

Multiplying by $\frac{r}{r^2-r_1^2}$ throughout & rearranging

$$\frac{d\sigma_r}{dr}\left(\frac{r^2}{r^2-r_1^2}\right) - \frac{2r_1^2 r}{r^2-r_1^2}\sigma_r = \frac{Tr}{(r^2-r_1^2)}$$

or $$\frac{d}{dr}\left(\frac{r^2\sigma_r}{r^2-r_1^2}\right) = \frac{Tr}{r^2-r_1^2}$$

By Integrating

$$\frac{r^2\sigma_r}{r^2-r_1^2} = \frac{T}{2}\log(r^2-r_1^2) + A \qquad (f)$$

where A is a constant of integration

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Let $r_3 = r_2 + d_w$

At $r = r_3$, $\sigma_r = 0$

$\therefore \quad A = -\frac{T}{2}\log(r_3^2 - r_1^2)$

$$\therefore \quad \frac{r^2\sigma_r}{r^2 - r_1^2} = \frac{T}{2}\log_e(r^2 - r_1^2) - \frac{T}{2}\log(r_3^3 - r_1^2)$$

$$\therefore \quad \boxed{(\sigma_r)_w = -\frac{T}{2}\left(\frac{r^2 - r_1^2}{r^2}\right)\log\left(\frac{r_3^2 - r_1^2}{r^2 - r_1^2}\right)} \quad (g)$$

where the subscript w stands for winding

By substituting equation (g) in (e)

$$(\sigma_\theta)_w = T - \frac{T}{2}\left(\frac{r^2 - r_1^2}{r^2}\right)\left(\frac{r^2 + r_1^2}{r^2 - r_1^2}\right)\log\left(\frac{r_3^2 + r_1^2}{r^2 - r_1^2}\right)$$

$$= T - \frac{T}{2}\left(\frac{r^2 + r_1^2}{r^2}\right)\log\left(\frac{r_3^2 - r_1^2}{r^2 - r_1^2}\right)$$

$$\boxed{(\sigma_\theta)_w = T\left[1 - \left[\frac{r^2 + r_1^2}{2r^2}\right]\log\left(\frac{r_3^2 - r_1^2}{r^2 - r_1^2}\right)\right]} \quad (h)$$

The values of winding on initial stresses in the cylinder is found by substituting $r = r_2$ in eqn (g)

$$\sigma_r = -\frac{T}{2}\left(\frac{r_2^2 - r_1^2}{r_2^2}\right)\log\left(\frac{r_3^2 - r_1^2}{r_2^2 - r_1^2}\right) = -p_2 \quad (i)$$

Eq. (i) may now be substituted in eqn (a) & (b) to determine initial stresses in the cylinder

$$\boxed{(\sigma_r)_c = -\frac{T}{2}\left(1 - \frac{r_1^2}{r^2}\right)\log\left(\frac{r_3^2 - r_1^2}{r_2^2 - r_1^2}\right)}$$

$$\boxed{(\sigma_\theta)_c = -\frac{T}{2}\left(1 + \frac{r_1^2}{r^2}\right)\log\left(\frac{r_3^2 - r_1^2}{r_2^2 - r_1^2}\right)}$$

where c stands for the cylinder

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Thick Spherical Shell :-

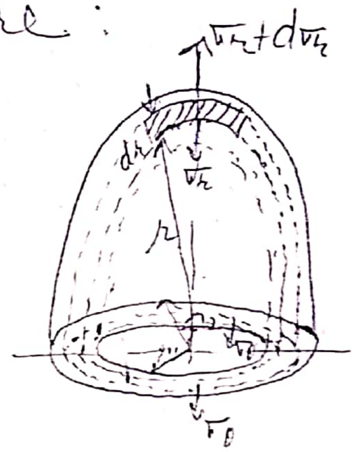
Consider a thick spherical shell of internal radius r_1 & external radius r_2 subjected to internal pressure p_1 & external pressure p_2.

Let

σ_r : Radial stress

σ_θ : Circumferential stress

Consider an elementary ring of the spherical shell between radii r & $r+dr$. The forces acting on the elementary ring are :



Upward force due to radial stresses $= \pi(r+dr)^2(\sigma_r + d\sigma_r) - \pi r^2 \sigma_r$

Downward force due to circumferential stresses

$= 2\pi r\, dr\, \sigma_\theta$

For equilibrium of the elementary ring

$$\pi(r+dr)^2(\sigma_r + d\sigma_r) - \pi r^2\sigma_r = 2\pi r\, dr\, \sigma_\theta$$

$$\pi(r^2+dr^2+2r\,dr)(\sigma_r+d\sigma_r) - \pi r^2\sigma_r = 2\pi r\, dr\, \sigma_\theta$$

$$\cancel{\pi r^2\sigma_r} + \underline{\pi dr^2\sigma_r} + \pi\, 2r\,dr\,\sigma_r + \underline{\pi dr^2 d\sigma_r} + \underline{\pi dr\, 2r\, d\sigma_r} + \pi r^2 d\sigma_r$$

$$- \cancel{\pi r^2 \sigma_r} = 2\pi r\, dr\, \sigma_\theta$$

Neglecting smaller quantities

$$\cancel{\pi}\, 2r\,\sigma_r\, dr + \cancel{\pi} r^2 d\sigma_r = 2\cancel{\pi}\, r\, dr\, \sigma_\theta$$

$$\sigma_\theta = \sigma_r \frac{2r\,dr}{2r\,dr} + \frac{d\sigma_r\, r^2}{2r\,dr}$$

$$\sigma_\theta = \sigma_r + \frac{r}{2}\frac{d\sigma_r}{dr} \quad (a)$$

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Differentiating (a) w.r.t r.

$$\frac{d\sigma_\theta}{dr} = \frac{d\sigma_r}{dr} + \frac{r}{2}\frac{d^2\sigma_r}{dr^2} + \frac{1}{2}\frac{d\sigma_r}{dr}$$

$$\frac{d\sigma_\theta}{dr} = \frac{3}{2}\frac{d\sigma_r}{dr} + \frac{r}{2}\frac{d^2\sigma_r}{dr^2} \qquad (b)$$

Let increase in radius r be u.

$$\therefore \varepsilon_\theta = \frac{2\pi(r+u) - 2\pi r}{2\pi r} = \frac{u}{r}$$

& Also $\varepsilon_\theta = \frac{1}{E}[\sigma_\theta - \nu(\sigma_r + \sigma_\theta)]$

$$\therefore \frac{u}{r} = \frac{1}{E}[\sigma_\theta - \nu(\sigma_r + \sigma_\theta)]$$

$$u = \frac{r}{E}[\sigma_\theta - \nu(\sigma_r + \sigma_\theta)]$$

Diff. w.r.t. r

$$\frac{du}{dr} = \frac{1}{E}[\sigma_\theta - \nu(\sigma_r + \sigma_\theta)] + \frac{r}{E}\left[\frac{d\sigma_\theta}{dr} - \nu\left(\frac{d\sigma_r}{dr} + \frac{d\sigma_\theta}{dr}\right)\right] \qquad (c)$$

Now, Radial strain $\varepsilon_r = \frac{du}{dr}$ {increase in dr be du}

& also $\varepsilon_r = \frac{1}{E}[\sigma_r - \nu(\sigma_\theta + \sigma_\theta]$

$$\therefore \frac{du}{dr} = \frac{1}{E}[\sigma_r - 2\nu\sigma_\theta] \qquad (d)$$

By comparing (c) & (d)

$$\frac{1}{E}[\sigma_r - 2\nu\sigma_\theta] = \frac{1}{E}[\sigma_\theta - \nu(\sigma_r + \sigma_\theta)] + \frac{r}{E}\left[\frac{d\sigma_\theta}{dr} - \nu\left(\frac{d\sigma_r}{dr} + \frac{d\sigma_\theta}{dr}\right)\right]$$

By substituting the values of σ_θ & $\frac{d\sigma_\theta}{dr}$ from eqn (a) & (b), we get

$$\sigma_r + \nu\sigma_r - 2\nu\sigma_\theta + \nu\sigma_\theta - \sigma_\theta = r\left[\frac{d\sigma_\theta}{dr}(1-\nu) - \nu\frac{d\sigma_r}{dr}\right]$$

$$(\sigma_r - \sigma_\theta)(1+\nu) = r\left[\frac{d\sigma_\theta}{dr}(1-\nu) - \nu\frac{d\sigma_r}{dr}\right]$$

Now, putting from (a) & (b)

$$\left(\cancel{\sigma_r} - \cancel{\sigma_r} - \frac{r\,d\sigma_r}{2\,dr}\right)(1+\nu) = r\left[\left(\frac{3}{2}\frac{d\sigma_r}{dr} + \frac{r}{2}\frac{d^2\sigma_r}{dr^2}\right)(1-\nu) - \nu\frac{d\sigma_r}{dr}\right]$$

$$-\frac{r\,d\sigma_r}{2\,dr} - \frac{\nu r\,d\sigma_r}{2\,dr} = \frac{r3}{2}\frac{d\sigma_r}{dr} + \frac{r^2}{2}\frac{d^2\sigma_r}{dr^2} - \frac{3}{2}\frac{r\,d\sigma_r}{dr}\nu - \frac{r^2}{2}\frac{d^2\sigma_r}{dr^2}\nu - \frac{\nu r\,d\sigma_r}{dr}$$

$$-\frac{r}{2}\frac{d\sigma_r}{dr} - \frac{3r}{2}\frac{d\sigma_r}{dr} = \frac{r^2}{2}\frac{d^2\sigma_r}{dr^2}(1-\nu) - \frac{5}{2}r\nu\frac{d\sigma_r}{dr} + \nu\frac{r}{2}\frac{d\sigma_r}{dr}$$

$$\cancel{\frac{r}{2}\frac{d\sigma_r}{dr}(1-\nu)r = \frac{r^2}{2}\frac{d^2\sigma_r}{dr^2}(1-\nu)}$$

$$-\frac{r^2}{2}\frac{d^2\sigma_r}{dr^2}(1-\nu) = -2r\nu\frac{d\sigma_r}{dr} + 2r\frac{d\sigma_r}{dr}$$

$$-\frac{r^2}{2}\frac{d^2\sigma_r}{dr^2}(\cancel{1-\nu}) = +2r\frac{d\sigma_r}{dr}(\cancel{1-\nu})$$

$$r\frac{d^2\sigma_r}{dr^2} + 4\frac{d\sigma_r}{dr} = 0$$

Let $x = \frac{d\sigma_r}{dr}$

$$\therefore r\frac{dx}{dr} + 4x = 0$$

$$\frac{dx}{x} = -4\frac{dr}{r}$$

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