TORSION OF CIRCULAR MEMBERS

Torsion :- A shaft is said to be in torsion, when equal & opposite torques are applied at the two ends of the shaft. The torque is equal to the product of force applied (tangentially to the ends of a shaft) & radius of the shaft.

When the bar is subjected to a torque so that bending is zero, then every section of the bar is subjected to a state of pure shear. The moment of resistance developed by shear stress being everywhere equal to the magnitude, & opposite in sense, to the applied torque.

Torsion of a circular shaft :-

ASSUMPTIONS :-

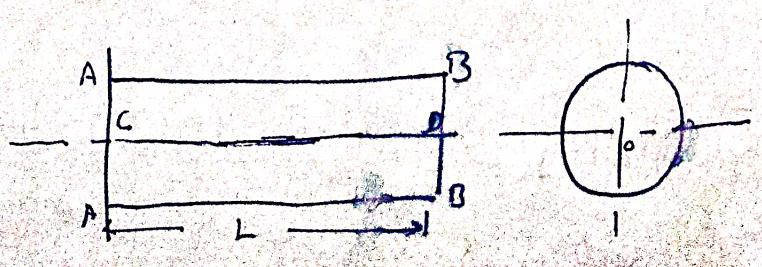
(a) The following assumptions are made in studying this problem:

(i) The material of the bar is homogeneous, perfectly elastic & obey Hook's law.

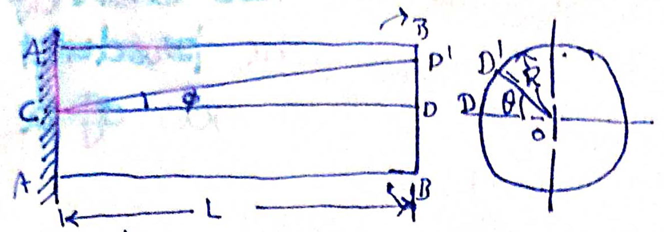
(ii) The stress does not exceed the limit of proportionality.

(iii) Cross-sections rotate as if rigid, i.e. every diameter rotates through same angle.

Derivation :- When a circular shaft is subjected to torsion, shear stresses are set up in the material of shaft. Now, consider a shaft fixed at end AA & free at the end BB. Let CD is any line on the outer



surface of the shaft. Now let the shaft is subjected to a torque T at the end B B.



The shaft at the end BB will rotate clockwise & stress at any point in the cross-section of the shaft is one of pure shear & the strain is such that one cross-section of the shaft moves relative to another. The point D will shift to D' & hence line CD will be deflected to CD' & line OD will be shifted to OD'.

Let R = Radius of shaft
L = Length of shaft
T = Torque applied at the end BB
f_s = Shear stress induced at the surface of the shaft due to torque T
G = Modulus of rigidity of the material of the shaft
ϕ = $\angle DCD'$ also equal to shear strain
θ = $\angle DOD'$ & is also called angle of twist.

$\therefore$ Shear strain at the outer surface

= Distortion per unit length = $\frac{DD'}{L}$

$\phi = \frac{DD'}{L}$ —(1)

Now, Arc $DD' = R \times \theta$ —(2) {Assume Arc $DD' \sim DD'$}

By putting the value of (2) in (1)

$\therefore \phi = \frac{R \times \theta}{L}$

& Modulus of rigidity $G = \frac{\text{Shear stress}}{\text{Shear strain}} = \frac{f_s}{\frac{R\theta}{L}}$

$\therefore \frac{G\theta}{L} = \frac{f_s}{R}$ —(3)

Now for a given shaft subjected to given torque (T), the value of C, θ & l are constant. Hence the shear stress produced is proportional to the radius R

$$f_s \propto R$$

$$\text{or } \frac{f_s}{R} = \text{Constant}$$

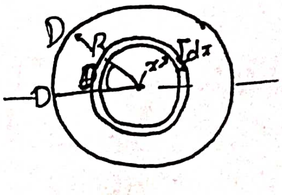
∴ At any radius x,

$$\frac{f_{sx}}{f_s} = \frac{x}{R}$$

$$\Rightarrow \boxed{f_{sx} = \frac{x}{R} \times f_s} \quad - (4)$$

$$\text{Also } \boxed{\frac{f_s}{R} = \frac{G\theta}{L}} \quad \text{from } (3)$$

→ Now consider an elementary ring of the shaft at a radius x & of thickness dx. The shear stress in the ring is f_{sx}



Total force on the ring

$$= \text{Area of ring} \times f_{sx}$$

$$= 2\pi x \, dx \times f_{sx}$$

Moment of this force about the axis of the shaft

$$= 2\pi x \, dx \cdot f_{sx} \cdot x$$

$$= 2\pi x^2 \, dx \cdot f_{sx}$$

$$= 2\pi x^2 \, dx \cdot \frac{x}{R} \times f_s \quad \{\because \text{from } (4)\}$$

$$= 2\pi x^3 \, dx \cdot \frac{f_s}{R}$$

Total resisting moment of the shaft cross-section is

$$= 2\pi \frac{f_s}{R} \int_0^R x^3\, dx$$

$$= 2\pi \frac{f_s}{R} \left| \frac{x^4}{4} \right|_0^R$$

$$= 2\pi \frac{f_s}{R} \left[\frac{R^4}{4} - 0 \right]$$

$$= \frac{\pi R^3}{2} f_s$$

But total resisting moment of the section

= Applied torque T

Torque transmitted by circular solid shaft $\rightarrow$ $\boxed{T = \frac{\pi R^3}{2} f_s} = \frac{\pi}{16} d^3 f_s$ $\{\because d/2 = R\}$

$$= \frac{f_s}{d/2} \cdot \frac{\pi d^4}{32} = \left[\frac{f_s}{(d/2)}\right] \cdot J$$

where $J = \frac{\pi d^4}{32}$ is the polar moment of inertia of the shaft cross-section

$$\boxed{\frac{T}{J} = \frac{f_s}{R}} \quad - (5)$$

From eqn. (3) & (5)

$$\boxed{\frac{T}{J} = \frac{f_s}{R} = \frac{G\theta}{L}}$$

KNOWN AS TORSION FORMULA FOR SHAFTS OF CIRCULAR CROSS-SECTION

Torque Transmitted by Hollow Shaft :-

Let d_o = Outer diameter of solid shaft & r_o be radius
d_i = Inner diameter of solid shaft & r_i be radius.

Proceed in similar way as in case of solid shaft

$$\therefore T_{HOLLOW} = 2\pi \frac{f_s}{r_o} \int_{r_i}^{r_o} x^3\, dx$$

$$= 2\pi \frac{f_s}{r_o} \left| \frac{x^4}{4} \right|_{r_i}^{r_o} = 2\pi \frac{f_s}{r_o} \left[\frac{r_o^4 - r_i^4}{4} \right]$$

$$= 2\pi \frac{f_s}{r_o} \cdot \frac{r_o^4 - r_i^4}{4} = \frac{f_s}{r_o} \cdot \frac{\pi}{2} \left[r_o^4 - r_i^4 \right]$$

$$\boxed{T_{HOLLOW} = \frac{f_s}{r_o} \cdot J_{HOLLOW}}$$

OR

$$\boxed{T_{HOLLOW} = \frac{f_s}{r_o} \cdot \frac{\pi}{2} \left[r_o^4 - r_i^4 \right]}$$

Comparison Of Hollow & Solid Shaft :-

(A) For The Same Outer Radius (i.e. $R = r_o$) & Shear Stress

$$T_{HOLLOW} = \frac{f_s}{r_o} \frac{\pi}{2} \left[r_o^4 - r_i^4 \right]$$

$$T_{SOLID} = \frac{f_s}{2} \cdot \pi R^3$$

$$\frac{T_{HOLLOW}}{T_{SOLID}} = \left(\frac{r_o^4 - r_i^4}{r_o} \right) \cdot \frac{1}{R^3} \quad -(1)$$

For, $R = r_o$

$$\frac{T_{HOLLOW}}{T_{SOLID}} = 1 - \frac{r_i^4}{r_o^4} = 1 - C^4 \quad \left\{ \text{where } C = \frac{r_i}{r_o} \right\}$$

& $C \leq 1$ $\therefore T_{HOLLOW} < T_{SOLID}$

$\therefore$ For same outer radius torsional strength of hollow shaft is less than solid shaft.

(3) For same amount of material & shear stress

Volume of solid shaft = Volume of hollow shaft

$$\pi R^2 L = \pi (r_o^2 - r_i^2) L$$

$$R^2 = (r_o^2 - r_i^2) \quad -②$$

$$\frac{T_{HOLLOW}}{T_{SOLID}} = \left(\frac{r_o^4 - r_i^4}{r_o}\right) \frac{1}{R^3} \quad \{\because \text{from } ①\}$$

$$= \frac{r_o^4 - r_i^4}{r_o (r_o^2 - r_i^2) R} \quad \{\text{from } ②\}$$

$$= \frac{\cancel{(r_o^2 - r_i^2)} (r_o^2 + r_i^2)}{r_o \cancel{(r_o^2 - r_i^2)} R}$$

$$= \frac{r_o^2 + r_i^2}{r_o R}$$

Now $C = \frac{r_i}{r_o}$

$$C^2 = \frac{r_i^2}{r_o^2} \Rightarrow r_i^2 = C^2 r_o^2$$

$$\& \quad \frac{C^2}{1 - C^2} = \frac{r_i^2}{r_o^2 - r_i^2} = \frac{r_i^2}{R^2} = \frac{C^2 r_o^2}{R^2}$$

$$\therefore \frac{r_o^2}{R^2} = \frac{\cancel{C^2}}{1 - C^2} \cdot \frac{1}{\cancel{C^2}} = \frac{1}{1 - C^2}$$

$$\frac{r_o}{R} = \frac{1}{\sqrt{1 - C^2}}$$

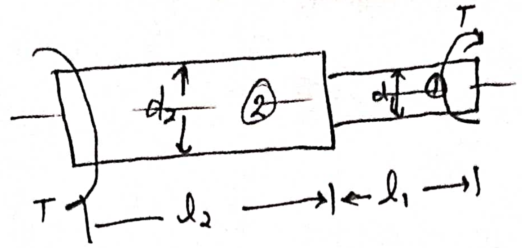
$$\therefore \frac{T_{HOLLOW}}{T_{SOLID}} = \frac{r_o^2 + r_i^2}{r_o R} = \frac{r_o^{\cancel{2}} \left[1 + \left(\frac{r_i}{r_o}\right)^2\right]}{\cancel{r_o} R} = \frac{1 + C^2}{\sqrt{1 - C^2}}$$

For $C < 1$, $T_{HOLLOW} > T_{SOLID}$

$\therefore$ For same amount of material & shear stress, the $\boxed{T_{HOLLOW} > T_{SOLID}}$

Composite Circular Shafts

(A) Shaft in series :-



Consider a shaft made of two materials connected in series & subjected to torque T. Since the both shafts are subjected to the same torque, hence

$$T_1 = T_2$$

from Torsion formula

$$\frac{T_1}{J_1} = \frac{fs_1}{d_1/2} = \frac{G_1 \theta_1}{l_1} \quad \therefore T_1 = \frac{J_1 fs_1}{\frac{d_1}{2}} = \frac{J_1 G_1 \theta_1}{l_1} \quad -(1)$$

& for section (2)

$$\frac{T_2}{J_2} = \frac{fs_2}{d_2/2} = \frac{G_2 \theta_2}{l_2} \quad \therefore T_2 = \frac{J_2 fs_2}{d_2/2} = \frac{J_2 G_2 \theta_2}{l_2} \quad -(2)$$

Dividing (1) by (2)

$$\frac{\cancel{T_1}}{\cancel{T_2}} = \frac{\frac{J_1 fs_1}{d_1/2}}{\frac{J_2 fs_2}{d_2/2}}$$

$$\therefore \frac{fs_1}{fs_2} = \frac{J_2}{J_1} \times \frac{d_1}{d_2} = \frac{\frac{\pi}{32} d_2^4}{\frac{\pi}{32} d_1^4} \times \frac{d_1}{d_2} \Rightarrow$$

$$\boxed{\frac{fs_1}{fs_2} = \left(\frac{d_2}{d_1}\right)^3}$$

$$\& \quad \frac{\theta_1}{\theta_2} = \frac{J_2 G_2}{l_2} \times \frac{l_1}{J_1 G_1}$$

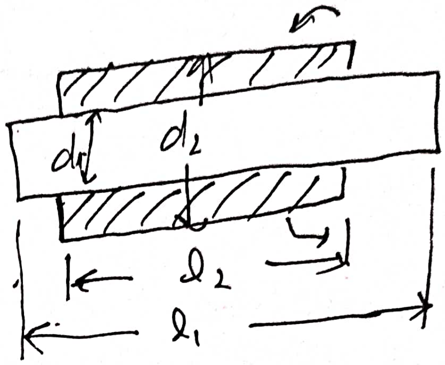
$$\boxed{\frac{\theta_1}{\theta_2} = \left(\frac{d_2}{d_1}\right)^4 \times \frac{l_1}{l_2} \times \frac{G_2}{G_1}}$$

(b) SHAFTS IN PARALLEL :- Now consider a shaft made of two materials connected in parallel. Let T be the torque applied to the composite shaft, then

$$T = T_1 + T_2$$

$$T_1 = \frac{G_1 \theta_1 J_1}{l_1}$$

$$T_2 = \frac{G_2 \theta_2 J_2}{l}$$



$$\therefore T = \frac{G_1 \theta_1 J_1}{l_1} + \frac{G_2 \theta_2 J_2}{l_2}$$

If $l_2 = l_1 = l$ & $\theta_1 = \theta_2 = \theta$, then

$$T = \frac{\theta}{l}(G_1 J_1 + G_2 J_2)$$ [For rigidly fixed or close fitted shaft]

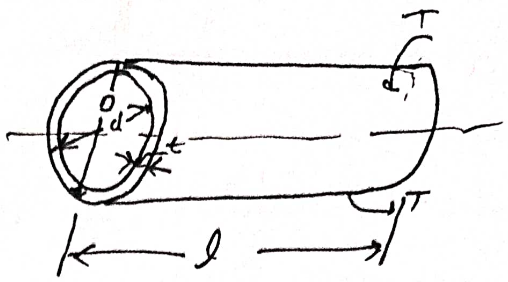
$$\theta = \frac{Tl}{G_1 J_1 + G_2 J_2}$$

where $J_1 = \frac{\pi d_1^4}{32}$

& $J_2 = \frac{\pi}{32}(d_2^4 - d_1^4)$

Torsion Of Thin Circular Tubes :-

Consider a thin circular tube subjected to torque T



Let t = Thickness of wall of tube

& $t = \frac{D-d}{2} \Rightarrow d = D-2t$

where D = outer diameter
d = Inner diameter
l = length of tube

& Polar moment of inertia

$$J = \frac{\pi}{32}(D^4-d^4) = \frac{\pi}{32}\left[D^4-(D-2t)^4\right]$$

$$= \frac{\pi}{32}\left[D^4 - D^4\left(1-\frac{2t}{D}\right)^4\right]$$

$$= \frac{\pi D^4}{32}\left[1-\left(1-\frac{8t}{D}+\cdots\right)\right]$$ [B] neglecting higher terms

$$= \frac{\pi D^4}{32}\left(\cancel{1}-\cancel{1}+\frac{8t}{D}\right)$$

$$J = \frac{\pi D^3 t}{4}$$

Now $\frac{T}{J} = \frac{f_s}{D/2} \Rightarrow T = \frac{f_s J}{D/2} = \frac{f_s \cdot \frac{\pi D^3}{4}t}{D/2}$

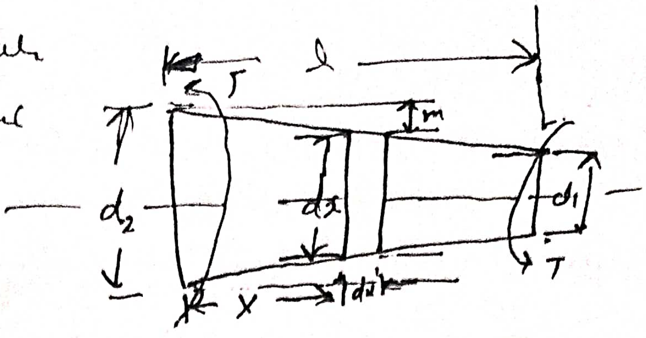
$$\therefore \boxed{T = \frac{\pi}{2}D^2 t f_s}$$

& $\frac{T}{J} = \frac{G\theta}{l}$

$$\Rightarrow \theta = \frac{Tl}{JG} = \frac{Tl}{\frac{\pi D^3}{4}tG}$$

$$\boxed{\theta = \frac{4Tl}{\pi D^3 tG}}$$ Angle of Twist

TAPERED CIRCULAR SHAFT :-



Consider a tapered circular shaft. Let T be the torque applied at the ends. Let f_{s_1} & f_{s_2} be the max. shear stress at the ends having diameter d_1 & d_2 respectively & f_{sx} be the shear stress at a distance x from the end having diameter d_2.

Since the Torque is same throughout the shaft

$$\frac{\pi}{16} d_1^3 f_{s_1} = \frac{\pi}{16} d_2^3 f_{s_2} = \frac{\pi}{16} d_x^3 f_{sx}$$

OR $d_1^3 f_{s_1} = d_2^3 f_{s_2} = d_x^3 f_{sx}$

Consider a small element of length dx of the shaft at a distance x from larger end. Let d_x be the diameter at this section.

If $d\theta$ = Angle of twist of small length dx

then $d\theta = \frac{T}{GJ} dx = \frac{32T}{\pi d_x^4 G} dx$

& $d_x = d_2 - \left(\frac{d_2-d_1}{l}\right)x$

$= d_2 - kx$ where $k = \frac{d_2-d_1}{l}$

$\therefore d\theta = \frac{32T\,dx}{\pi G (d_2-kx)^4}$

Total angle of twist $\theta = \int_0^l \frac{32T\,dx}{\pi G (d_2-kx)^4} = \frac{32T}{\pi G}\int_0^l (d_2-kx)^{-4} dx$

$$= \frac{32T}{\pi G} \left| (d_2-kx)^{-3} \right|_0^l \cdot \frac{1}{3k}$$

$$= \frac{32T}{\pi G} \left[(d_2-kl)^{-3} - d_2^{-3} \right] \frac{1}{3k}$$

$$= \frac{32Tl}{3\pi G (d_2-d_1)} \left[(d_2-d_2+d_1)^{-3} - d_2^{-3} \right]$$

$$\boxed{\theta = \frac{32Tl}{3\pi G} \left(\frac{d_1^2+d_2^2+d_1 d_2}{d_1^3 d_2^3} \right)} \quad \& \quad \boxed{f_{s_{max}} = \frac{16T}{\pi d_1^3}}$$

Combined effect of Axial, Bending & Torsional stresses

Consider a body under the combined effect of axial, bending & torsional stresses. Then

Direct stress, $\sigma_d = P/A$

P = Axial applied load

A = Area of cross-section

Bending stress, $\sigma_b = \frac{My}{I}$

where M = Bending moment

y = Distance of fibre from the neutral axis

I = Moment of inertia

Torsional stress

$\tau = \frac{Tr}{J}$

T = Torque

r = radius of shaft

J = Polar moment of inertia

The principal stresses under the combined action of these stresses become,

$$\sigma_{1,2} = \frac{\sigma_d \pm \sigma_b}{2} \pm \frac{1}{2}\sqrt{(\sigma_d \pm \sigma_b)^2 + 4\tau^2}$$

& Max. shear stress $\tau_{max} = \frac{1}{2}\sqrt{(\sigma_d \pm \sigma_b)^2 + 4\tau^2}$

For a circular shaft of diameter d,

$$\sigma_d = \frac{4P}{\pi d^2}$$

$$\sigma_b = \frac{32M}{\pi d^3}$$

$$\tau = \frac{16T}{\pi d^3}$$

$$\sigma_{1,2} = \frac{1}{2}\left(\frac{4P}{\pi d^2} \pm \frac{32M}{\pi d^3}\right) \pm \frac{1}{2}\sqrt{\left(\frac{4P}{\pi d^2} \pm \frac{32M}{\pi d^3}\right)^2 + \left(\frac{32T}{\pi d^3}\right)^2}$$

$$= \frac{16}{\pi d^3}\left(\frac{Pd}{8} \pm M\right) \pm \frac{16}{\pi d^3}\sqrt{\left(\frac{Pd}{8} \pm M\right)^2 + T^2}$$

$$= \frac{16}{\pi d^3}\left[\left(\frac{Pd}{8} \pm M\right) \pm \sqrt{\left(\frac{Pd}{8} \pm M\right)^2 + T^2}\right]$$

$$\tau_{max} = \frac{16}{\pi d^3}\sqrt{\left(\frac{Pd}{8} \pm M\right)^2 + T^2}$$

If $P = 0$, then

$$\sigma_{1,2} = \frac{16}{\pi d^3}\left[M \pm \sqrt{M^2+T^2}\right]$$

$$= \frac{32}{\pi d^3}\left\{\frac{1}{2}\left(M \pm \sqrt{M^2+T^2}\right)\right\}$$

$$= \frac{32}{\pi d^3} M_e$$

Where $M_e = \frac{1}{2}\left(M + \sqrt{M^2+T^2}\right)$, equivalent bending moment may be defined as the bending moment which will produce the same max. direct stress as produced by the bending moment & torque acting simultaneously.

$\tau_{max} = \frac{16}{\pi d^3}\sqrt{M^2+T^2} = \frac{16}{\pi d^3} T_e$, equivalent torque may be defined as the torque which will produce the same max. shear stress as produced by the bending moment & torque acting simultaneously.

Polar Modulus :- Polar modulus of a shaft section is the ratio of polar moment of inertia of the shaft section to the maximum radius.

$$\text{Polar Modulus} = \frac{J}{r}$$

Torsional rigidity :- Torsional rigidity of shaft is the product of modulus of rigidity ~~&~~ polar moment of inertia (J)

$$\text{Torsional rigidity} = G \times J$$

It is ~~a~~ torque required to produce a twist of one radian per unit length of the shaft.

* If shaft are in series

& Torque is applied at the joint

then $T = T_1 + T_2$