Geometric Transformations

2D & 3D

With the procedure for displaying output primitives & their attributes, we can create a variety of pictures & graphs. In many applications, there is also a need for altering & manipulating displays. Design applications & facility layouts are created by arranging the orientations & sizes of the component parts of the scene. And animations are produced by moving the 'camera' or the objects in a scene along animation paths. Changes in orientation, size & shape are accomplished with geometric transformations that alter the coordinate descriptions of objects. The basic geometric transformations ~~that alter the coordinates~~ are translation, rotation & scaling. Other transformations that are often applied to objects include reflection & shear.

Translation

A translation is applied to an object by repositioning it along a straight-line path from one coordinate location to another. We translate a two-dimensional point by adding translation distances, t_x & t_y to the original coordinate position (x, y) to move the point to a new position (x', y')

$$x' = x + t_x$$
$$y' = y + t_y$$

The translation distance pair (t_x, t_y) is called a translation vector or shift vector.

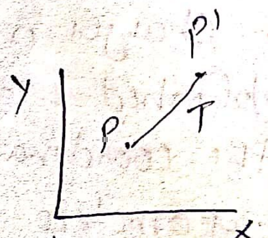
In matrix form

$$\begin{bmatrix} x_1' \\ y_1' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

Or
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$P' = T(t_x, t_y) \cdot P$$

where 1 is homogeneous coordinate



Translating a point from position P to position P' with translation vector T.

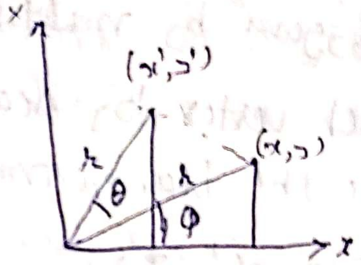
Translation is a rigid-body transformation that moves objects without deformation. That is, every point on the object is translated by the same amount.

Rotation :-

A 2D-rotation is applied to an object by repositioning it along a circular path in the xy plane. To generate a rotation, we specify a rotation angle θ & the position (x_r, y_r) of the rotation point (or pivot point) about which the object is rotated. Positive values for the rotation angle define counterclockwise rotation about the pivot point & the negative values rotate the object in the clockwise direction.

This transformation can also be described as rotation about a rotation axis that is perpendicular to the xy plane & passes through the pivot point. ③

$x' = r\cos(\phi+\theta)$
$y' = r\sin(\phi+\theta)$



$x' = r\cos\phi\cos\theta - r\sin\phi\sin\theta$
$y' = r\cos\phi\sin\theta + r\sin\phi\cos\theta$

$x = r\cos\phi$ & $y = r\sin\phi$

$\therefore\ x' = x\cos\theta - y\sin\theta$
$y' = x\sin\theta + y\cos\theta$

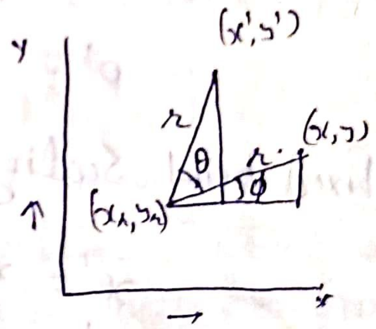
$\therefore\ P' = R.P$

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

R is called rotation matrix.

Rotation of a point about an arbitrary point position is given as

$x' = x_r + (x - x_r)\cos\theta - (y - y_r)\sin\theta$
$y' = y_r + (x - x_r)\sin\theta + (y - y_r)\cos\theta$



~~Scali~~ In matrix notion

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$P' = R(\theta).P$

Scaling :-

A scaling transformation alters the size of an object. This operation can be carried out for polygons by multiplying the coordinate values (x, y) of each vertex by scaling factors S_x & S_y to produce the transformed coordinates (x', y')

$$x' = x\,S_x \quad , \quad y' = y \cdot S_y$$

Scaling factor S_x scale object in the x direction, while S_y scales in the y direction.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$P' = S \cdot P$$

where S is the 2 by 2 scaling matrix.

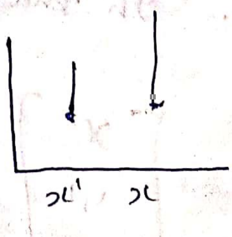
or

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$P' = S(S_x, S_y) \cdot P.$$

Fixed point Scaling :

$$x' = x_f + (x - x_f) S_x$$
$$y' = y_f + (y - y_f) S_y$$



$$x' = x \cdot S_x + x_f(1 - S_x)$$
$$y' = y \cdot S_y + y_f(1 - S_y)$$

$S_x = S_y$ Uniform Scaling
$S_x \neq S_y$ Differential Scaling

an angle α with HP & inclined

Homogeneous Coordinates:-

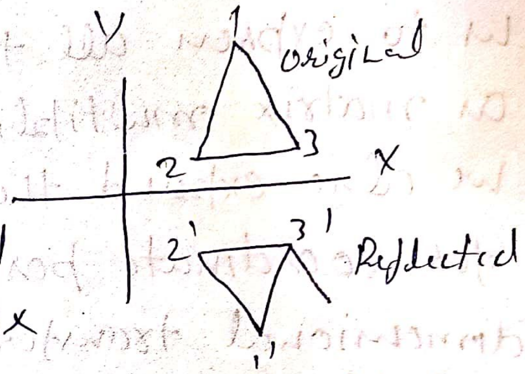
We can combine the multiplication & translational term for two-dimensional geometric transformations into a single matrix representation by expanding the 2x2 matrix representation to 3 by 3 matrices. This allows us to express all the transformation equations as matrix multiplications, providing that we also expand the matrix representation for coordinate positions. To express any two dimensional transformation as a matrix multiplication, we represent each Cartesian coordinate position (x, y) with the homogeneous (triple) coordinate (x_h, y_h, h) where

$$x = \frac{x_h}{h}, \quad y = \frac{y_h}{h}$$

Thus, a general homogeneous co-ordinate representation can also be written as $(h.x, h.y, h)$. For two-dimensional geometric transformations, we can choose the homogeneous parameter h to be any nonzero value. Thus, there is an infinite number of equivalent homogeneous representations for each coordinate point (x, y). A convenient choice is simply to set $h = 1$. Each two-dimensional position is then represented with homogeneous coordinates $(x, y, 1)$.

Reflection :- A reflection is a transformation that produces a mirror image of an object. The mirror image for two dimensional object is generated relative to an axis of reflection by rotating the object 180° about the reflection axis.

Reflection about the line $y = 0$ the x-axis, is accomplished with the transformation matrix

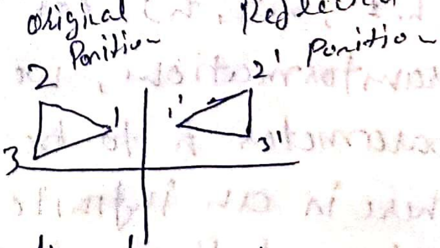


$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This transformation keeps x values the same, but 'flips' the y value of coordinate positions.

A reflection about y axis flips x coordinates while keeping y coordinates the same. The matrix for this transformation is

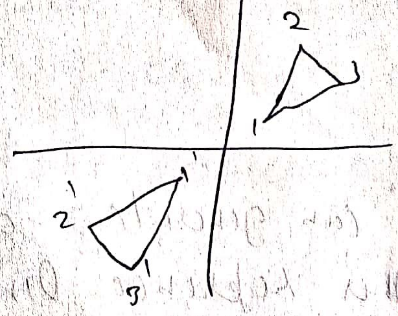
$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



We flip both the x & y coordinates of a point by reflecting relative to an axis that is perpendicular to the xy plane & that passes through the coordinate origin. This transformation, referred to as a an angle α with HP & inclined

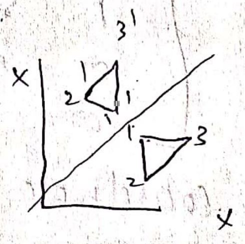
Reflection relative to the coordinate origin. the matrix representation:

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



For line $y = x$
Transformation matrix

$$\begin{bmatrix} 0 & +1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



<u>Shear</u>:- A transformation that distorts the shape of an object such that the transformed shape appears as if the object were composed of internal layers that had been caused to slide over each other is called a shear. Two common shearing transformations are those that shift coordinate x values & those that shift y values.

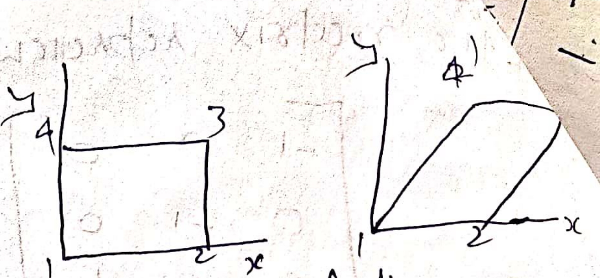
An x-direction shear relative to the x-axis is produced the transformation matrix

$$\begin{bmatrix} 1 & sh_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which transforms coordinate positions as

$$x' = x + sh_x \cdot y \qquad y' = y$$

Any real number can be assigned to the shear parameter Sh_x.



We can generate x-direction shears relative to other reference lines with

$$\begin{bmatrix} 1 & Sh_x & -Sh_x \cdot y_{ref} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with coordinate position transformed as

$$X \begin{bmatrix} 1 & Sh_x & -Sh_x \cdot y_{ref} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

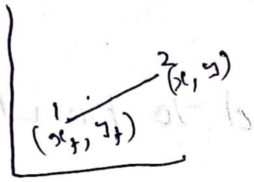
$$x' = x + Sh_x (y - y_{ref}), \quad y' = y$$

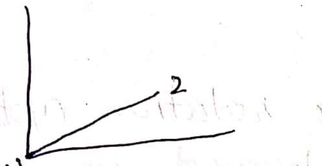
Composite Transformations :-

With the matrix representations, we can set up a matrix for any sequence of transformations as a composite transformation matrix by calculating the matrix product of the individual transformations. Forming products of transformation matrices is often referred to as a concatenation or composition, of matrices. For column matrix representation of co-ordinate positions, we form composite transformations by multiplying matrices in order from right to left. That is, each successive transformation matrix premultiplies the product of preceding transformation matrix

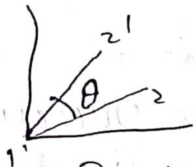
Fixed or pivot point Rotation :-

1. Translate the object so that the pivot-point position is moved to the co-ordinate origin.
2. Rotate the object about the coordinate origin.
3. Retranslate the object so that the pivot-point is returned to its original position.

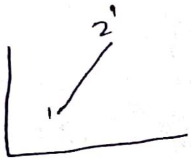




Translate to origin



Rotate at given angle θ



Retranslate to original positi

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & x_f \\ 0 & 1 & y_f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_f \\ 0 & 1 & -y_f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$P' = T(x_f, y_f) \quad R(\theta) \quad T(-x_f, -y_f) \quad P$$

Rotation about fixed point

In this similar way, we can do scaling.

Translation :-

Composite transformation matrix for the sequence of translation is

$$\begin{bmatrix} 1 & 0 & tx_2 \\ 0 & 1 & ty_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & tx_1 \\ 0 & 1 & ty_1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & tx_1 + tx_2 \\ 0 & 1 & ty_1 + ty_2 \\ 0 & 0 & 1 \end{bmatrix}$$

Which demonstrates that two successive transformation are additive.

Rotation :-

Two successive rotation applied to point P produce the transformed position

$$P' = R(\theta_2)\{R(\theta_1) . P\}$$

$$= \{R(\theta_2)\ R(\theta_1)\} . P$$

$$\therefore R(\theta_2) . R(\theta_1) = R(\theta_1 + \theta_2)$$

$$\therefore P' = R(\theta_1 + \theta_2) . P$$

Scaling :-

Concatenating transformation matrices for two successive scaling operation

$$\begin{bmatrix} Sx_2 & 0 & 0 \\ 0 & Sy_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Sx_1 & 0 & 0 \\ 0 & Sy_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} Sx_1 . Sx_2 & 0 & 0 \\ 0 & Sy_1 . Sy_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or $S(Sx_2 . Sy_2) . S(Sx_1 . Sy_1) = S(Sx_1 . Sx_2 , Sy_1 . Sy_2)$

Q) object about an axis which (15)

Transformations :-

(11)

Method of for geometric transformation & object modeling in 3D are extented from 2D methods by including considerations for the z-coordinate.

Translation :-

In 3D homogeneous coordinate representation, a point is translated from position $P = (x, y, z)$ to position $P' = (x', y', z')$ with the matrix operation

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$P' = T.P$$

Parameters t_x, t_y, t_z specifying translation distances for the co-ordinate directions x, y, & z are assigned any real values.

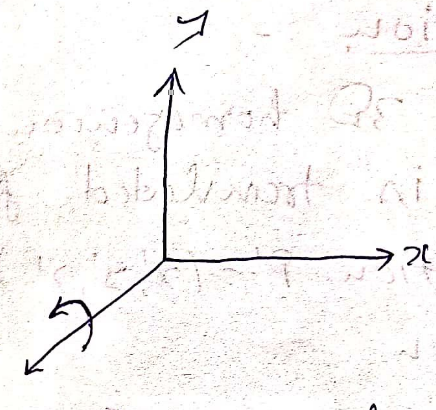
Rotation :-

To generate a rotation transformation for an object, we must designate an axis of rotation (about which the object is to be rotated) & the amount of angular rotation. Unlike 2D application, where all transformation are carried out in xy plane, a 3D rotation can be specified around any line in space. The easiest rotation axes to handle are those that are parallel to the coordinate axes.

(i) Co-ordinate-Axes Rotation

The 2D z-axis rotation equation are easily extended to 3D.

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$
$$z' = z$$



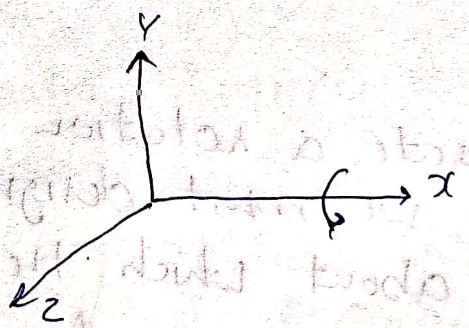
Parameter θ specifies the rotation angle. In homogeneous co-ordinate form, the 3D z-axis rotation equations are expressed as

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$P' = R_z(\theta)\, P$$

X-axis rotation :-

$$y' = y\cos\theta - z\sin\theta$$
$$z' = y\sin\theta + z\cos\theta$$
$$x' = x$$



$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

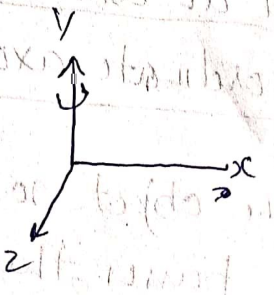
$$P' = R_x(\theta)\, P$$

... about an axis which
axis rotation

$z' = z\cos\theta - x\sin\theta$

$x' = z\sin\theta + x\cos\theta$

$y' = y$

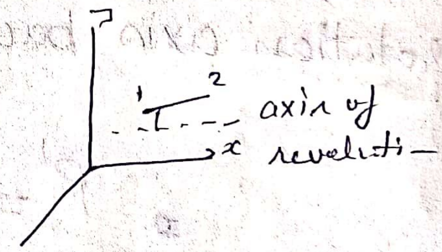


$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$P' = R_y(\theta)\, P$$

Rotation about an axis which is parallel to one of the coordinate axis

1. Translate the object so that the rotation axis coincides with the ~~parallel~~ parallel coordinate axis

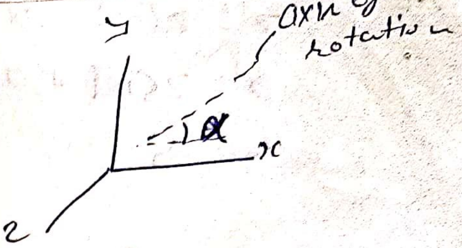


2. Perform the specified rotation about that axis
3. Translate the object so that the rotation axis is moved back to its original position

$$P' = T^{-1} R_x(\theta)\, T\, P$$

where the composite matrix for transformation is $R(\theta) = T^{-1} R_x(\theta)\, T$

(iii) Rotation about an axis that is not parallel to one of the coordinate axes or inclined at an angle α to x-axis.

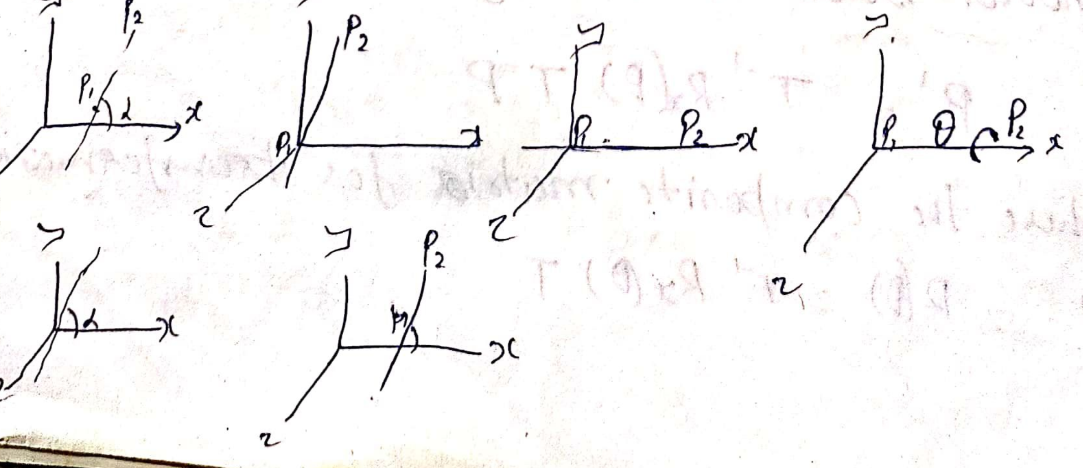


1. Translate the object so that the rotation axis passes through the coordinate origin.
2. Rotate the object so that the axis of rotation coincides with one of the coordinate axis.
3. Perform the specified rotation about that coordinate axis.
4. Apply inverse rotations to bring the rotation axis back to its original position.
5. Apply the inverse translation to bring the rotation axis back to its original position.

$$P' = T^{-1} R_x^{-1}(\alpha) R_x(\theta) R_x(\alpha) T \, P$$

$$\therefore R(\theta) = T^{-1} R_x^{-1}(\alpha) R_x(\theta) R_x(\alpha) T$$

$$\boxed{P' = R(\theta) P}$$



Rotation of object(θ) about an axis which is inclined at an angle α with HP & inclined at an angle β with V.P

1. Translate the object so that the rotation axis passes through the coordinate origin.
2. Rotate the object by an angle α about x-axis so that axis of rotation comes in a plane.
3. Rotate the axis by an angle β about y-axis so that axis of rotation coincides with z-axis
4. Perform the specified rotation about that coordinate axis.
5. Apply inverse rotation to bring the rotation axis back by β angle.
6. Apply inverse rotation by angle α to bring the rotation axis back to its original orientation.
7. Apply the inverse translation to bring the rotation axis back to its original position.

$$R(\theta) = T^{-1} R_x^{-1}(\alpha) R_y^{-1}(\beta) R_z(\theta) R_y(\beta) R_x(\alpha) T$$

$$\boxed{P' = R(\theta) P}$$

Reflections :-

A 3D reflection can be performed relative to a selected reflection axis or with respect to a selected reflection plane. In general, 3D reflection matrices are set up similarly to those for two dimensions. Reflections relative to a given axis are equivalent to 180° rotation about that axis.

The matrix representation for this reflection of points relative to the xy plane is

$$RF_z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Shears :- Shearing Transformations can be used to modify the object shapes. They are also useful in 3D viewing for obtaining general projection transformations.

As an example of 3D shearing, the following transformation produces a z-axis shear.

$$SH_z = \begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Parameter a & b can be assigned any real value. The effect of this transformation matrix is to alter x & y coordinate values by an amount that is proportional to the z value, while leaving the z-coordinate unchanged