Surfaces

A surface model of an object is more complete & less ambiguous representation than its wireframe model. It is also richer in its associated geometric contents, which makes it more suitable for engineering & design applications. Surface models take the modeling of an object one step beyond the wireframe models by providing information on surfaces connecting the object edges. Typically a surface model consists of wireframe entities that form the basis to create surface entities. Surface description is usually tackled as an extension to the wireframe representation. Analytic & synthetic surface entities are available & provided by most CAD/CAM systems.

Applications :-

Shape design & representation of complex objects such as car, ship & airplane bodies as well as castings cannot be achieved utilizing wireframe modeling. In such cases, surface modeling must be utilized to describe object precisely & accurately.

Due to richness in information of surface models their use in engineering & design ~~department~~ environments can be extended beyond just geometric design & representation. They are usually used in various applications such as calculating mass properties, checking for interference between mating parts, generating cross-sectioned views, generating finite element meshes & generating NC tool paths for continuous path machining.

Difference b/w Surface Models & Solid Models surface.

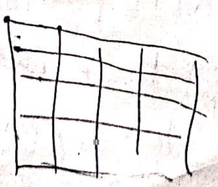
Surface models define only the geometry of their corresponding objects. They store no information regarding the topology of these objects.

To create a surface model, the user begins by constructing wireframe entities & then connecting them appropriately with proper surface entities. For solids based on boundary representation, either faces, edges & vertices are created or solid primitives are input which are converted internally by the software to faces, edges & vertices.

Surface Entities :- There are two types of entities: Analytic & Synthetic surface entities.

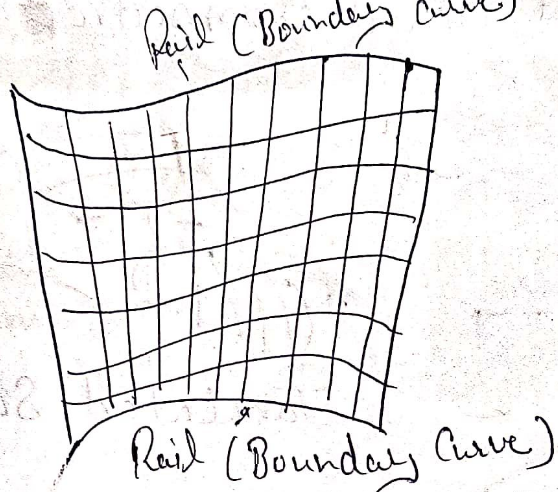
Analytic surface entities include plane surface, ruled surface, surface of revolution & tabulated cylinder. Synthetic entities include the bicubic Hermite spline surface, B-spline surface, rectangular & triangular, Bezier patches, rectangular & triangular coon patches & Gordon surface.

1. Plane Surface :- This is the simplest surface. It requires three non coincident points to define infinite plane. The plane surface can be used to generate cross-sectional views by intersecting a surface model with it, generate cross-sections for mass property calculations or other similar applications where a plane is needed.



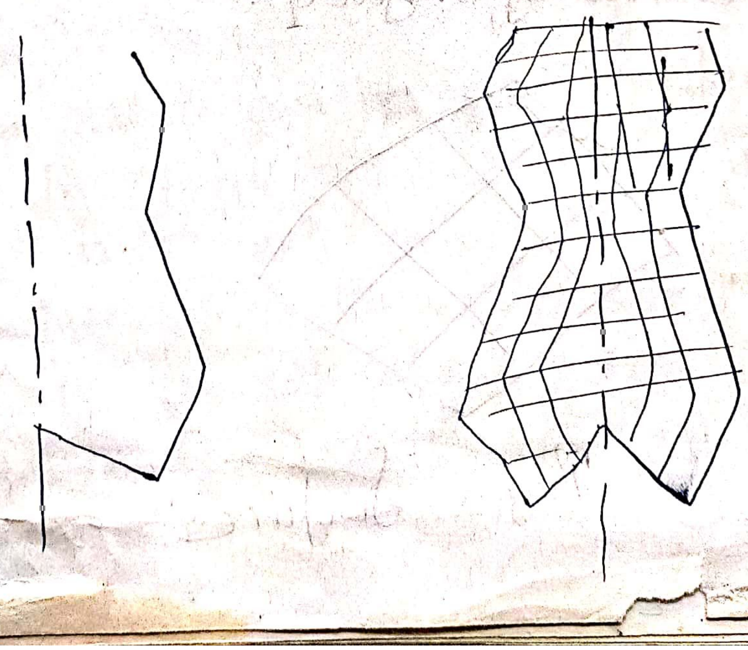
one plane   Plane Surface.

Ruled (Lofted Surface) :- This is a linear surface. It interpolates linearly between two boundary curves that defines the surface (rails). Rails can be wireframe entity. This entity is ideal to represent surfaces that do not have any twists or kinks.



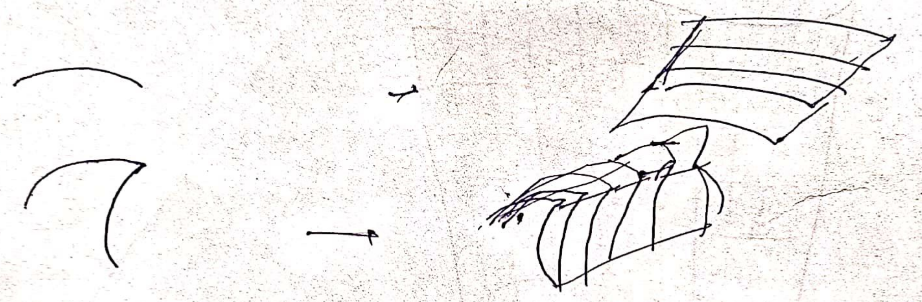
Ruled Surface

3. **Surface of Revolution** :- There is an axisymmetric surface that can be model axisymmetric objects. It is generated by rotating a planar wireframe entity in space about the axis of symmetry at a certain angle.



4. Tabulated Cylinder :-

This is a surface generated by translating a planar curve a certain distance along a specified direction (axis of the cylinder). The plane of the curve is perpendicular to the axis of the cylinder. It is used to generate surfaces that have identical curved cross section.

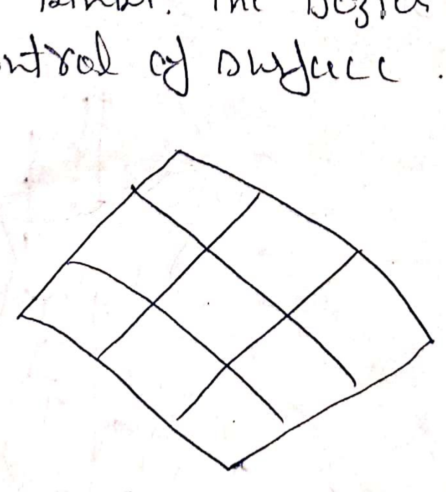


Curve — Tabulated Surface

5. Bezier Surface :-

This is a surface that approximates given input data. It is different from the previous surfaces in that it is a synthetic surface. Similar to the Bezier curve, it does not pass through all given data points. It is a general surface that permits twists & kinks. The Bezier surface allows only global control of surface.

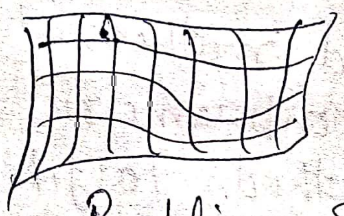
6.



Bezier Surface

B-spline Surface :-

This is a surface that can approximates given input data. It is a synthetic surface. It is a general surface but with the advantage of permitting local control of the surface.



B-spline Surface

7. Coons Patch :-

The coons patch is used to create a surface using curves that form the closed boundaries.

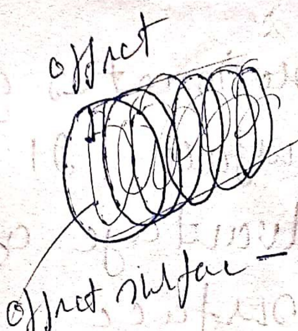
Coon Patch

8. Fillet Surface :

This is a B-spline surface that blends two surfaces together. The two original surfaces may or may not be trimmed.

9. Offset surface :-

Existing surface can be offset to create new ones identical in shape but may have different dimensions. It is a useful surface to use to speed up surface construction. Offset surface command is very efficient to use when the original surface is a composite one.



Surface Representation :-

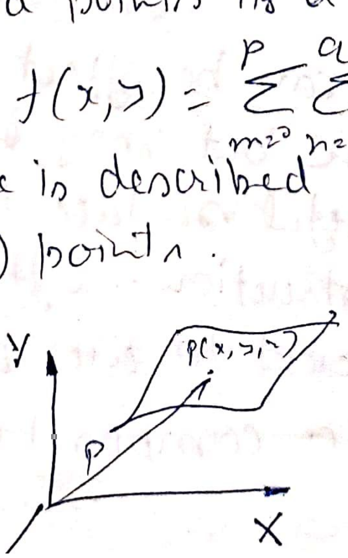
Surface can be described mathematically in three dimensional space by nonparametric or parametric equations. There are several methods to fit non-parametric surfaces to a given set of data points. These fall in to two categories. In the first, one equation is fitted to pass through all the points while in the second the data points are used to develop a series of surface patches that are connected together with at least position & first derivative continuity. In both cases, the equation of the surface or surface patch is given by

$$P = [x, y, z]^T = [x, y, f(x,y)]^T$$

where P is the position vector of a point on the surface. The natural ~~function~~ form of the function $f(x,y)$ for a surface to pass through all the given data points is a polynomial, that is,

$$z = f(x,y) = \sum_{m=0}^{p} \sum_{n=0}^{q} a_{mn} x^m y^n$$

where the surface is described by an xy grid of size $(p+1) \times (q+1)$ points.



Point P on a Nonparametric Surface Patch.

The parametric representation of a surface means a continuous, vector-valued function $P(u,v)$ of two variables, or parameters, u & v where the variables are allowed to range over some connected region of the uv plane. The function $P(u,v)$ at certain u & v values is the point on the surface at these values. The most general way to describe the parametric equation of a 3D curved surface in space is

$$P(u,v) = [x \ y \ z]^T$$

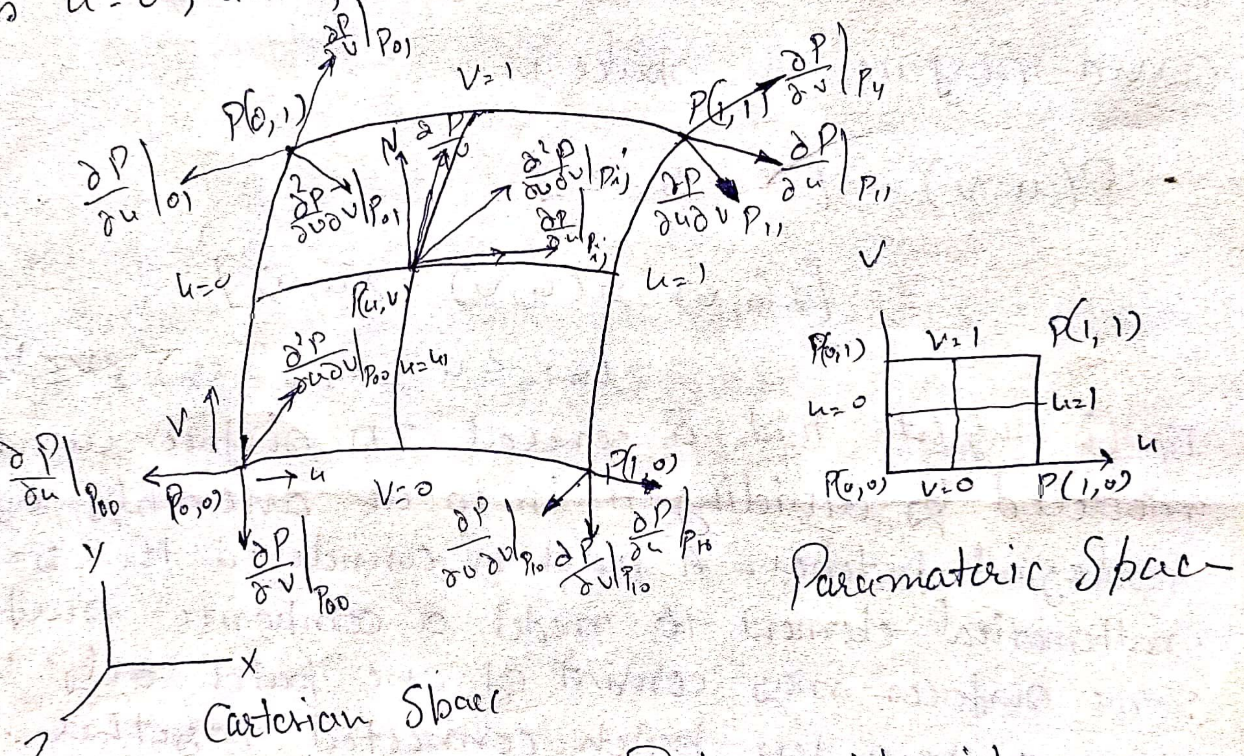
$$= [x(u,v) \ y(u,v) \ z(u,v)]^T$$

$$u_{min} \leq u \leq u_{max}, \ v_{min} \leq v \leq v_{max} \quad (1)$$

Eqn (1) suggests that a general 3D surface can be modeled by dividing it into an assembly of topological patches. A patch is considered the basic mathematical element to model a composite surface. Some surfaces may consist of one patch only while other may be a few patch connected together. The topology of patch may be rectangular or triangular. Triangular patches add more flexibility in surface modeling because they do not require ordered rectangular array of data points to create the surface as the rectangular patches do.

Parametric Representation of a 3D-Surface:-

There is a set of boundary conditions associated with rectangular patch. There are sixteen vectors & four boundary curves. The vectors are four position vectors for the four corner point $P(0,0)$, $P(1,0)$, $P(1,1)$ & $P(0,1)$; eight tangent vectors (two at each corner) & four twist vectors at the corner points. The four boundary curves are described by holding one parametric variable fixed at one of its limit values & allowing the other to change freely. The boundary curves are then defined by the curve equations $u=0$, $u=1$, $v=0$ & $v=1$



A Parametric Surface Patch with its Boundary Conditions

The tangent vector at any point $P(u,v)$ on the surface is obtained by holding one parameter constant & differentiating with respect to the other. Therefore, there are two tangent vectors, a tangent to each of the intersecting curves passing through the point. These vectors are given by

$$P_u(u,v) = \frac{\partial P}{\partial u} = \frac{\partial x}{\partial u} i + \frac{\partial y}{\partial u} j + \frac{\partial z}{\partial u} k \qquad u_{min} \le u \le u_{max}$$

$$v_{min} \le v \le v_{max}$$

along v = constant curve &

$$P_v(u,v) = \frac{\partial P}{\partial v} = \frac{\partial x}{\partial v} i + \frac{\partial y}{\partial v} j + \frac{\partial z}{\partial v} k$$

along u = Constant curve. These two equation can be combined to give

$$\begin{bmatrix} P_u \\ P_v \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{bmatrix}$$

Tangent vectors are useful in determining boundary condition for patching surfaces together as well as defining the motion of the cutters along the surface during maching process. The magnitude & unit vectors of the tangent vectors are given by

$$|P_u| = \sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2}$$

$$P_v = \sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2}$$

& $\hat{n}_u = \frac{P_u}{|P_u|} \qquad \hat{n}_v = \frac{P_v}{|P_v|}$

The twist vector at a point on a surface is said to measure the twist in surface at the point. It is the rate of change of tangent vector P_u with respect to v or P_v with respect to u, or it is the cross derivative vector at the point. If we increment u & v by Δu & Δv respectively & draw the tangent vectors, the incremental change in P_u & P_v at point P, whose position vector is $P(u,v)$, are obtained by translating $P_u(u, v+\Delta v)$ & $P_v(u+\Delta u, v)$ to P & forming the two triangles.

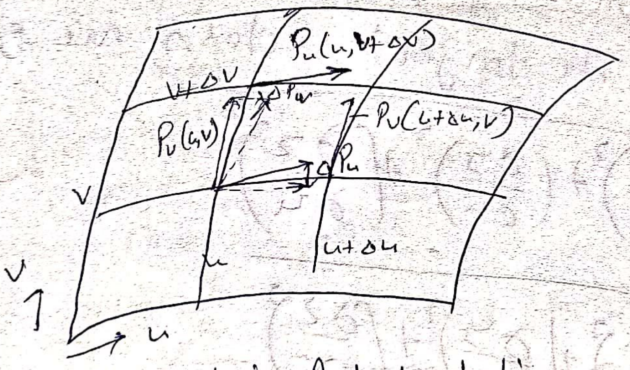
$$\lim_{\Delta v \to 0} \frac{\Delta P_u}{\Delta v} = \frac{\partial P_u}{\partial v} = \frac{\partial^2 P}{\partial u \partial v} = P_{uv}$$

$$\lim_{\Delta u \to 0} \frac{\Delta P_v}{\Delta u} = \frac{\partial P_v}{\partial u} = \frac{\partial^2 P}{\partial u \partial v} = P_{uv}$$

The twist vector can be written in terms of its cartesian components as

$$P_{uv} = \left[\frac{\partial^2 x}{\partial u \partial v} \; \frac{\partial^2 y}{\partial u \partial v} \; \frac{\partial^2 z}{\partial u \partial v}\right]^T = \frac{\partial^2 x}{\partial u \partial v}\hat{i} + \frac{\partial^2 y}{\partial u \partial v}\hat{j} + \frac{\partial^2 z}{\partial u \partial v}\hat{k}$$

$$u_{min} \le u \le u_{max}, \quad v_{min} \le v \le v_{max}$$



Geometric Interpretation of Twist vectors

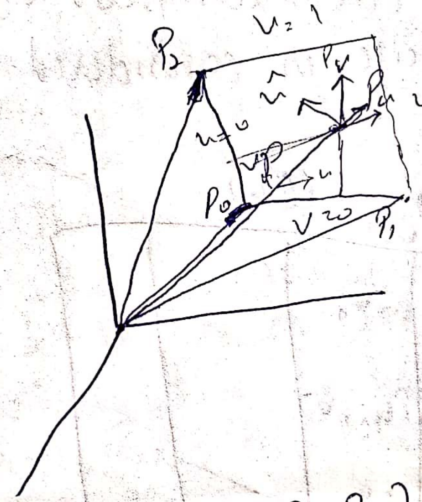
Parametric Representation of Analytical Surfaces:-

1. Plane Surface:-

The parametric equation of a plane can take different forms depending on the given data. Consider first the case of a plane defined by three points P_0, P_1 & P_2 as shown

Assume that the point P_0 defines $u=0$ & $v=0$ & vectors (P_1-P_0) & (P_2-P_0) define the u & v directions respectively. Assume also that the domain for u & v are $[0,1]$. The position vectors at any point P on the plane can be now written as

$$P(u,v) = P_0 + u(P_1-P_0) + v(P_2-P_0) \quad 0\le u\le 1,\ 0\le v\le 1 \quad (1)$$



$$P(u,v) = P_0 + u(P_1-P_0) + v(P_2-P_0)$$

The tangent vector at point P are

$$\frac{\partial P(u,v)}{\partial u} = P_1-P_0 \qquad \frac{\partial P(u,v)}{\partial v} = P_2-P_0 \qquad 0\le u\le 1,\ 0\le v\le 1$$

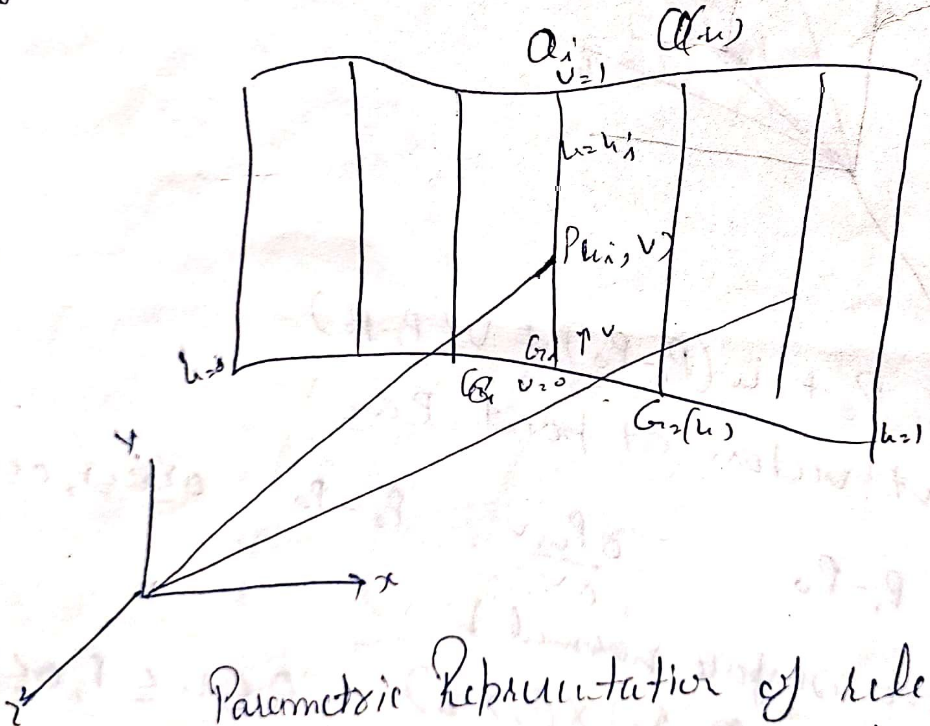
& the surface normal n

$$\hat{n}(u,v) = \frac{(P_1-P_0)\times(P_2\times P_0)}{|(P_1-P_0)\times(P_2 * P_0)|} \qquad 0\le u\le 1,\ 0\le v\le 1$$

which is constant for any point on the plane. As for the curvature of the plane, it is equal to zero, because all the second fundamental co-efficients of the plane are zero.

Ruled Surface :

A ruled surface is generated by joining corresponding points on two space curves $G(u)$ & $Q(u)$ by straight lines (also called rulings or generators). The main characteristic of a ruled surface is that there is at least one straight line passing through the point $P(u,v)$ & lying entirely in the surface. In addition every developable surface is a ruled surface. Cones & Cylinders are examples of ruled surfaces & the planer surface is considered the simplest of the ruled surfaces.



Parametric Representation of ruled surface.

To develop parametric representation of ruled surface, consider the ruling $u=u_i$ joining points G_i & Q_i on the rail $G(u)$ & $Q(u)$

$$P(u,v) = G_i + v(Q_i - G_i) \quad (1)$$

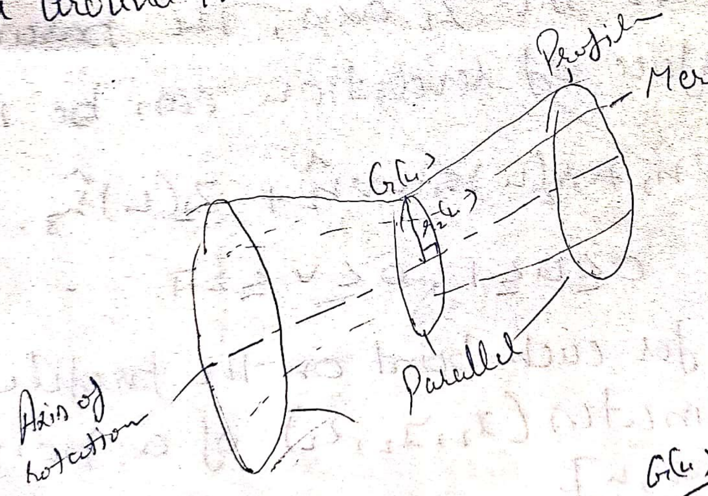
where v is the parameter along the ruling. Generalizing eqn (1) for any ruling, the parametric equation of a ruled surface defined by two rails in

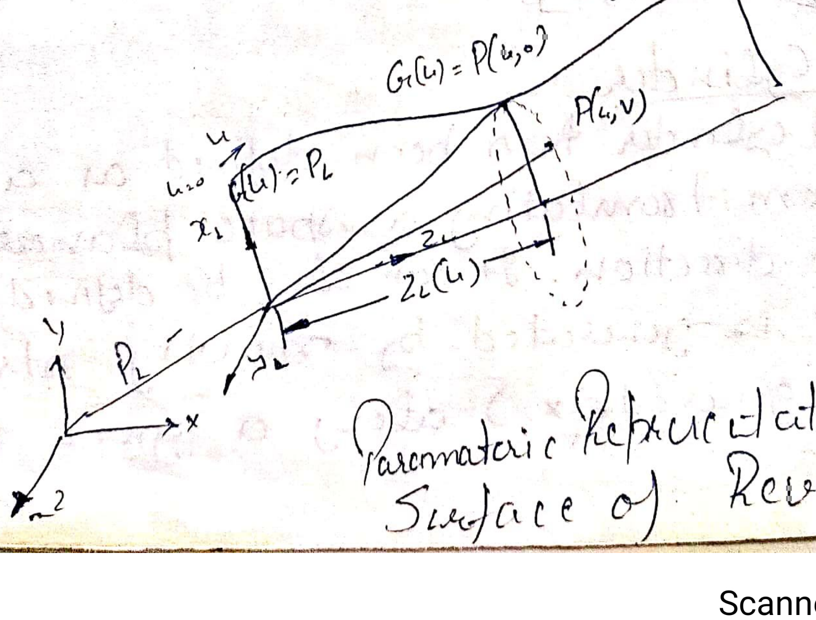
$$P(u, v) = G(u) + v\left[Q(u) - G(u)\right]$$
$$= (1-v)\, G(u) + v\, Q(u) \qquad 0 \le u \le 1,\ 0 \le v \le 1 \quad (2)$$

Holding the u value constant in above equation produces the rulings given by eqn (1) in the v direction of surface, while holding the v value constant yields the curves in the u direction which are a linear blend of the rails.

3. Surface of Revolution :-

The rotation of a planar curve an angle v about an axis of rotation creates a circle (if v = 360) for each point on the curve whose center lies on the axis of rotation & whose radius $r_z(u)$ is variable. The planar curve & the circles are called the profile & parallel respectively while the various positions of the profile around the axis are called meridians.





Paramataric Representation of Surface of Revolution

The planar curve & the axis of rotation form the plane of zero angle, that is, $v=0$. To derive the parametric equation of a surface of revolution, a local coordinate system with Z axis coincident with the axis of rotation. This local system shown by subscript L can be created as follows.
Choose the perpendicular direction from point $u=0$ on the profile as the X_L axis & the intersection point between X_L & Z_L as the origin of the local system. The Y_L axis is automatically determined by right-hand rule. Now, consider a point $G(u) = P(u,0)$ on the profile that rotates an angle v about Z_L when the profile rotates the same angle. Considering the shaded triangle which is perpendicular to the Z_L axis, the parametric equation of the surface of revolution can be written

$$P(u,v) = r_z(u)\cos v\,\hat{n}_1 + r_z(u)\sin v\,\hat{n}_2 + Z_L(u)\,\hat{n}_3,$$

$$0 \le u \le 1,\ 0 \le v \le 2\pi \quad -(1)$$

If we choose $Z_L(u) = u$ for each point on the profile, eq (1) gives the local coordinates (x_L, y_L, z_L) of a point $P(u,v)$ as $[r_z(u)\cos v, r_z(u)\sin v, u]$.

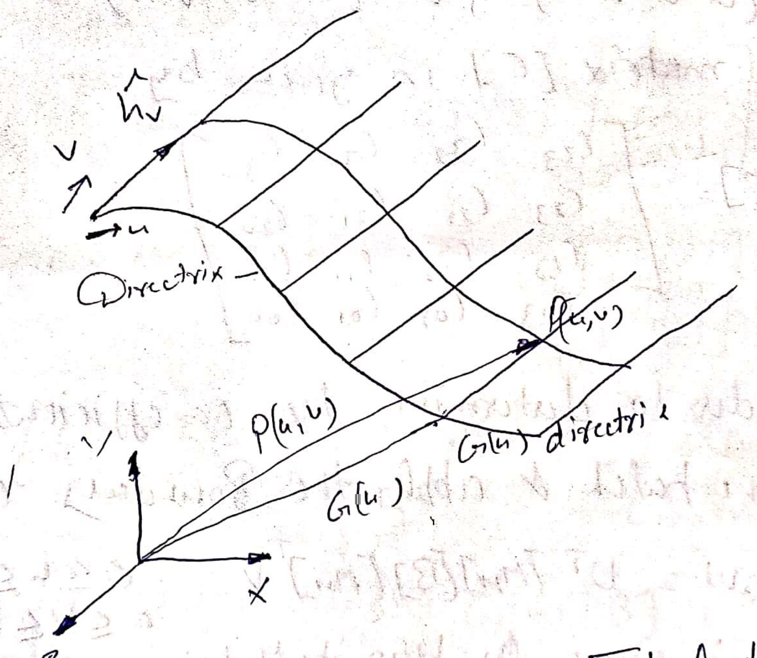
4. Tabulated Cylinder :-

A tabulated cylinder has been defined as a surface that results from translating a space planar curve along a given direction. It can also be defined as a surface that is generated by moving straight line (called generatrix) along a given planar

...ve (called directrix). The straight line always stays parallel to a fixed given vector that defines the v direction of the cylinder. The planar curve $G(u)$ can be any wireframe entities. The position vector of any point $P(u,v)$ on the surface can be written as

$$P(u,v) = G(u) + v\hat{n}_v \qquad 0 \le u \le u_{max}, \; 0 \le v \le v_{max} \qquad (1)$$

From a user point of view, $G(u)$ is the desired curve the user digitizes to form the cylinder, v is the cylinder length & $\hat{n}$ is the cylinder axis.



Parametric Representation of a Tabulated Cylinder

Parametric Representation of Synthetic Surface

Hermite Bicubic Surface :-

The parametric bicubic surface patch connects four corner data points & utilizes a bicubic equation. Therefore, 16 vector conditions (or 48 Scalar Conditions) are required to find the coefficients of the equ... When these coefficients are the four corner data points, the eight tangent vectors at the corner points, the Hermite bicubic surface patch results. The bicubic equation can be written as

$$P(u,v) = \sum_{i=0}^{3} \sum_{j=0}^{3} C_{ij} u^i v^j, \quad 0 \le u \le 1, \; 0 \le v \le 1 \qquad (1)$$

In matrix form

$$P(u,v) = U^T [C] V \qquad 0 \le u \le 1, \; 0 \le V \le 1$$

where $U = [u^3 u^2 u \; 1]^T$, $V = [v^3 \; v^2 \; v \; 1]^T$ & the coefficient matrix $[C]$ is given by

$$[C] = \begin{bmatrix} C_{33} & C_{32} & C_{31} & C_{30} \\ C_{23} & C_{22} & C_{21} & C_{20} \\ C_{13} & C_{12} & C_{11} & C_{10} \\ C_{03} & C_{02} & C_{01} & C_{00} \end{bmatrix}$$

In order to determine the co-efficients C_i, consider the patch & apply the Boundary condition

$$P(u,v) = U^T [M_H][B][M_H]^T V \qquad 0 \le u \le 1, \; 0 \le V \le 1 \qquad (2)$$

where M_H is given as Hermite Matrix & $[B]$ the geometry or boundary condition matrix is

$$B = \left[\begin{array}{cc|cc} P_{00} & P_{01} & P_{v00} & P_{v01} \\ P_{10} & P_{11} & P_{v10} & P_{v11} \\ \hline P_{u00} & P_{u01} & P_{uv00} & P_{uv01} \\ P_{u10} & P_{u11} & P_{uv10} & P_{uv11} \end{array}\right]$$

$$= \left[\begin{array}{c|c} [P] & P_v \\ \hline P_u & P_{uv} \end{array}\right]$$

where $[P]$, $[P_u]$, $[P_v]$ & $[P_{uv}]$ are the submatrices of the corner points, corner u-tangent vectors & corner v tangent vectors & the corner twist vectors respectively.

The tangent & twist vectors at any point on surface are given by

$$P_u(u,v) = U^T [M_H]^u [B] [M_H]^T V \quad (3)$$

$$P_v(u,v) = U^T [M_H] [B] [M_H]^{vT} V \quad (4)$$

$$P_{uv}(u,v) = U^T [M_H]^u B [M_H]^{vT} V \quad (5)$$

Expanding eqn. (2), (3), (4) & (5)

$$P(u,v) = [F_1(u)\ F_2(u)\ F_3(u)\ F_4(u)]\ [B] \begin{bmatrix} F_1(v) \\ F_2(v) \\ F_3(v) \\ F_4(v) \end{bmatrix}$$

Tangent

$$P_u(u,v) = [G_1(u)\ G_2(u)\ G_3(u)\ G_4(u)]\ [B] \begin{bmatrix} F_1(v) \\ F_2(v) \\ F_3(v) \\ F_4(v) \end{bmatrix}$$

$$P_v(u,v) = [F_1(u)\ F_2(u)\ F_3(u)\ F_4(u)]\ [B] \begin{bmatrix} G_1(v) \\ G_2(v) \\ G_3(v) \\ G_4(v) \end{bmatrix}$$

Tangent

$$P_{uv}(u,v) = [G_1(u)\ G_2(u)\ G_3(u)\ G_4(u)]\ [B] \begin{bmatrix} G_1(v) \\ G_2(v) \\ G_3(v) \\ G_4(v) \end{bmatrix}$$

where
$F_1(x) = 2x^3 - 3x^2 + 1$ $\quad P_0$
$F_2(x) = -2x^3 + 3x^2$ $\quad P_1 \quad$ — (7)
$F_3(x) = x^3 - 2x^2 + x$ $\quad P_0'$
$F_4(x) = x^3 - x^2$ $\quad P_1'$

&
$G_1(x) = 6x^2 - 6x$ $\quad$ — (8) $\quad$ (Tangent)
$G_2(x) = -6x^2 + 6x$
$G_3(x) = 3x^2 - 4x + 1$
$G_4(x) = 3x^2 - 2x$

For $u=0$ & $u=1$ edges these equation becomes

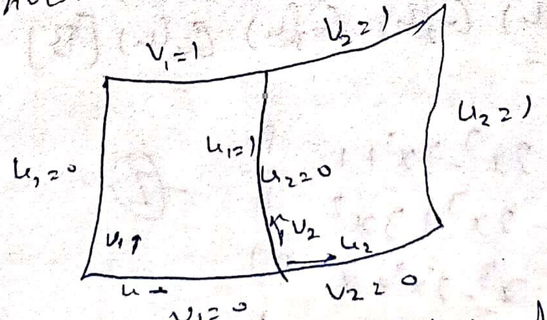
$$\begin{bmatrix} P(0,v) \\ P(1,v) \\ P_u(0,v) \\ P_u(1,v) \end{bmatrix} = [B] \begin{bmatrix} F_1(v) \\ F_2(v) \\ F_3(v) \\ F_4(v) \end{bmatrix} \quad (9)$$

Similarly, for $v=0$ & $v=1$ edges, we can write

$$\begin{bmatrix} P(u,0) \\ P(u,1) \\ P_v(u,0) \\ P_v(u,1) \end{bmatrix} = [B] \begin{bmatrix} F_1(u) \\ F_2(u) \\ F_3(u) \\ F_4(u) \end{bmatrix} \quad (10)$$

Eqn (10) shows that the corner v-tangent & twist vectors affect the position & tangent vectors respectively all along the $u=0$ & $u=1$ edges except at $v=0$ & 1 where F_3 & F_4 are both zero.

To write the blending conditions for C' continuity between two patches, consider the surface as shown here.



These conditions to connect patch 1 & patch 2 along the u edges are

$$[P(0,v)]_{Patch\,2} = [P(1,v)]_{Patch\,1} \qquad C^0 \text{ continuity}$$
$$[P_u(0,v)]_{Patch\,2} = k[P_u(1,v)]_{Patch\,1} \qquad C' \text{ continuity} \quad (11)$$

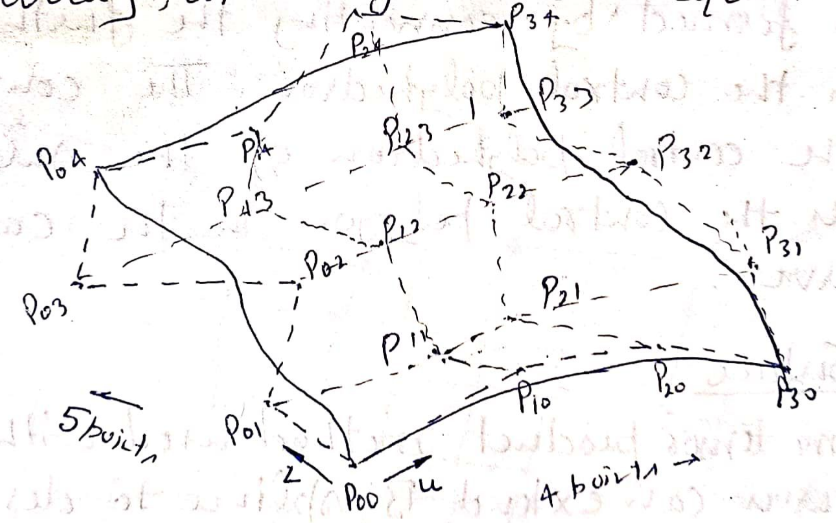
where k is constant

Bezier Surface :-

A tensor product Bezier surface is an extension of Bezier curve in two parametric directions u & v. An orderly set of data or control points is used to build a topologically rectangular surface. The surface equation can be written as

$$P(u,v) = \sum_{i=1}^{n} \sum_{j=1}^{m} P_{ij}\, B_{i,n}(u)\, B_{j,m}(v) \qquad 0 \le u \le 1,\ 0 \le v \le 1 \quad (1)$$

where $P(u,v)$ is any point on the surface & P_{ij} are the control points. These points form the vertices of the control or characteristic polyhedron of the resulting Bezier surface. The points are arranged in an $(n+1) \times (m+1)$ rectangular array, as seen from above equation.



4 X 5 Bezier Surface

Expanding the above equation (1)

$$P_{(u,v)} = P_{00}(1-u)^n (1-v)^m + P_{11}\, C(n,1)\, C(m,1)\, uv\, (1-u)^{n-1} (1-v)^{m-1}$$
$$+ P_{22}\, {}^{n}C_2\, {}^{m}C_2\, u^2 v^2 (1-u)^{n-2} (1-v)^{m-2} + \cdots$$
$$+ P_{(n-1)(m-1)}\, {}^{n}C_{n-1}\, {}^{m}C_{m-1}\, u^{n-1} v^{m-1} (1-u)(1-v) + P_{nm} u^n v^m$$

Symmetric terms in u & v

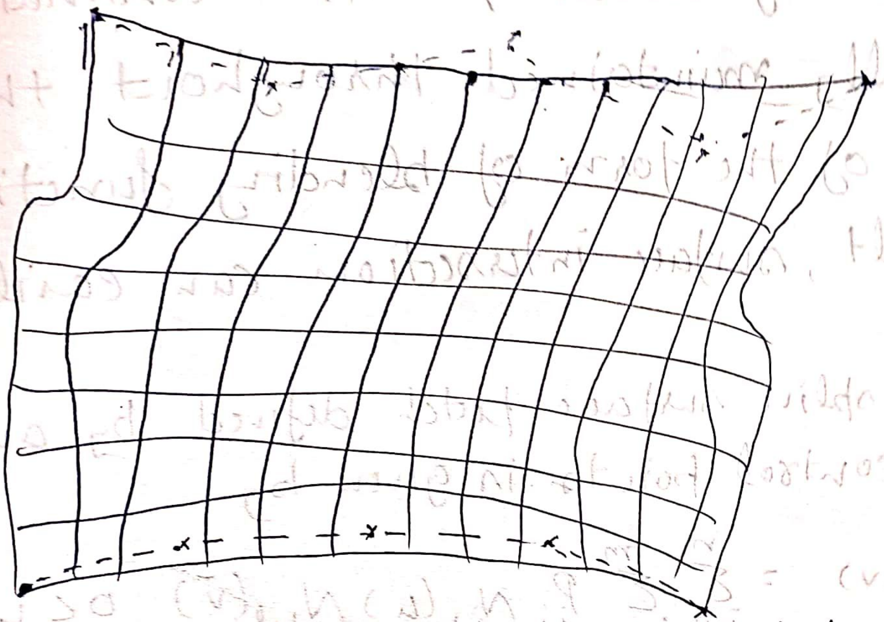
Non Symmetrical in terms of u & v

$$
\left[\begin{aligned}
&+ P_{10}\,{}^{n}C_1\, u(1-u)^{n-1}(1-v)^m + P_{01}\,{}^{m}C_1\, v(1-u)^n(1-v)^{m-1} \\
&+ P_{20}\,{}^{n}C_2\, u^2(1-u)^{n-2}(1-v)^m + P_{02}\,{}^{m}C_2\, v^2(1-u)^n(1-v)^{m-2} \\
&+ \ldots + P_{n0}\, u^n (1-v)^m + P_{0m}\, v^m (1-u)^m \\
&+ \ldots + P_{(n-1)(m-2)}\,{}^{n}C_{n-1}\,{}^{m}C_{m-2}\, u^{n-1} v^{m-2} (1-u)(1-v)^2 \\
&+ P_{(n-2)(m-1)}\,{}^{n}C_{n-2}\,{}^{m}C_{m-1}\, u^{n-2} v^{m-1} (1-u)^2 (1-v) \\
&+ P_{(n-1)m}\,{}^{n}C_{n-1} u^{n-1} v^m (1-u) + P_{n(m-1)}\, u^n v^{m-1} (1-v) \\
&+ P_{(n-1)m}\,{}^{n}C_{n-1}\, u^{n-1} v^m (1-u) + P_{n(m-1)}\, u^n v^{m-1} (1-v)
\end{aligned}\right. \qquad (2)
$$

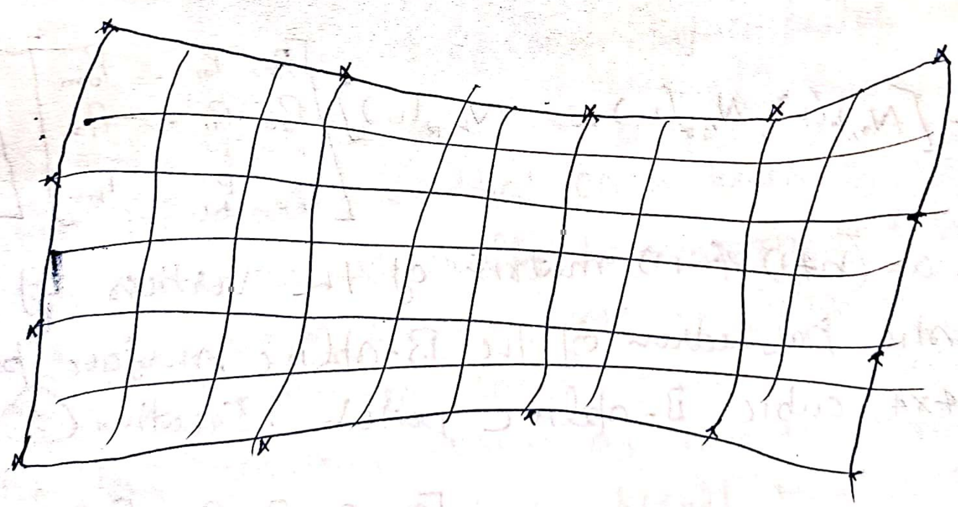
The characteristics of the Bezier surface are the same as those of the Bezier curve. The surface is also tangent to the corner segments of control polyhedron. — In addition, the Bezier surface possesses the convex hull property. The convex hull in this case is the polyhedron formed by connecting the furthest control points on the control polyhedron. The convex hull includes the control polyhedron of the surface as it includes the control polygon in the case of Bezier curve.

B-spline Surface :

The same tensor product method used with Bezier ~~surfaces~~ curves can extend B-splines to describe B-spline surfaces. A rectangular set of data (control) points creates the surface. This set forms the vertices of the characteristic polyhedron that approximates & control the shape of the resulting surface. A B-spline surface can approximate or interpolate the vertices of the polyhedron.



Patch approximates data points



Patch interpolates data points

4 X 5 B-spline Surface Patches

The degree of the surface is independent of the number of control points & continuity is automatically maintained throughout the surface by virtue of the form of blending functions. As a result, surface intersection can easily be managed.

A B-spline surface patch defined by an $(n+1)\times(m+1)$ array of control points is given by

$$P(u,v) = \sum_{i=0}^{n} \sum_{j=0}^{m} P_{ij}\, N_{i,k}(u)\, N_{j,l}(v) \quad 0 \le u \le u_{max} \quad (1)$$
$$0 \le v \le v_{max}$$

The major advantage of B-spline surface is the local control. Composite B-spline surfaces can be generated with C^0 & or C^1 continuity in the same way as composite Bezier surfaces.

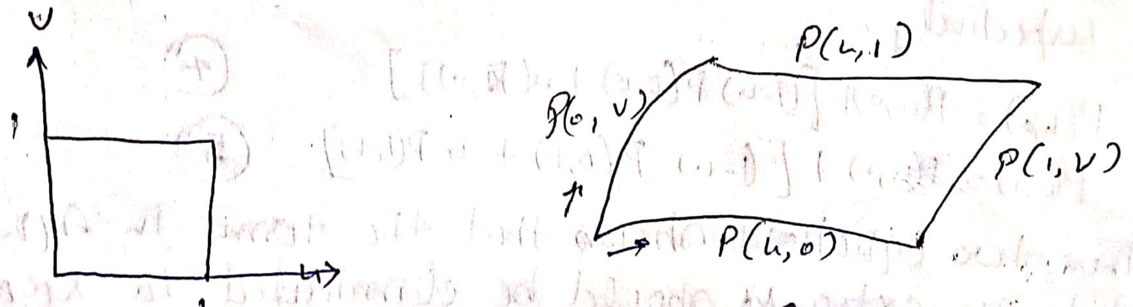
$$P(u,v) = [N_{0,k}(u), N_{1,k}(u) \ldots N_{n,k}(u)] \begin{bmatrix} P_{00} & P_{01} & \ldots & P_{0m} \\ P_{10} & P_{11} & \ldots & P_{1m} \\ \vdots & \vdots & & \\ P_{n0} & P_{n1} & \ldots & P_{nm} \end{bmatrix} \begin{bmatrix} N_{0,l}(v) \\ N_{1,l}(v) \\ \vdots \\ N_{m,l}(v) \end{bmatrix} \quad (2)$$

$[P]$ is an $(n+1)\times(m+1)$ matrix of the vertices of the characteristic polyhedron of the B-spline surface patch. For a 4x4 cubic B-spline patch, Equation (2) becomes

$$P(u,v) = [N_{0,4}(u)\ N_{1,4}(u)\ N_{2,4}(u)\ N_{3,4}(u)] \begin{bmatrix} P_{00} & P_{01} & P_{02} & P_{03} \\ P_{10} & P_{11} & P_{12} & P_{13} \\ P_{20} & P_{21} & P_{22} & P_{23} \\ P_{30} & P_{31} & P_{32} & P_{33} \end{bmatrix} \begin{bmatrix} N_{0,4}(v) \\ N_{1,4}(v) \\ N_{2,4}(v) \\ N_{3,4}(v) \end{bmatrix}$$

...on Surfaces :-

Coons surface patch is a form of "transfinite interpolation" which indicates that the Coons scheme interpolates to an infinite numbers of data points, that is, to all points of a curve segment, to generate the surface. The coons patch is particularly useful in blending four prescribed intersecting curves which form a closed boundary. The figure shown that the given four boundary curves as $P(u,0)$, $P(1,v)$, $P(u,1)$ & $P(0,u)$.



Parametric Space   Cartesian Space

Boundaries of Coons Surface patch

It is assumed that u & v range from 0 to 1 along their boundaries & that each pair of opposite boundary curves are identically parametrized. ~~Development of the Coons surface~~ Let us first consider the case of a bilinearly blended Coons patch which interpolates to the four boundary curves as shown above. For this case, it is useful to recall that a ruled surface interpolates linearly between two given boundary curves in one direction. Therefore, the superposition of two ruled surfaces connecting the two pairs of boundary curves might satisfy the boundary curve conditions & produces the coons patch.

$\therefore P_1(u,v) = (1-u)P(0,v) + u P(1,v)$ (1)

& $P_2(u,v) = (1-v)P(u,0) + v P(u,1)$ (2)

Adding these two equations gives the surface

$P(u,v) = P_1(u,v) + P_2(u,v)$ (3)

The resulting surface patch described by eqn (3) does not satisfy the boundary conditions.

For example, substituting $v=0$ & 1 into this equation gives respectively

$P(u,0) = P(u,0) + [(1-u)P(0,0) + u(P(1,0))]$ (4)

$P(u,1) = P(u,1) + [(1-u)P(0,1) + u P(1,1)]$ (5)

These two equations shows that the terms in square brackets are extra & should be eliminated to recover the original boundary curve. These terms define the boundaries of an unwanted surface $P_3(u,v)$ which is embedded in eqn (3). This surface can be defined by linear interpolation in the v direction, that is,

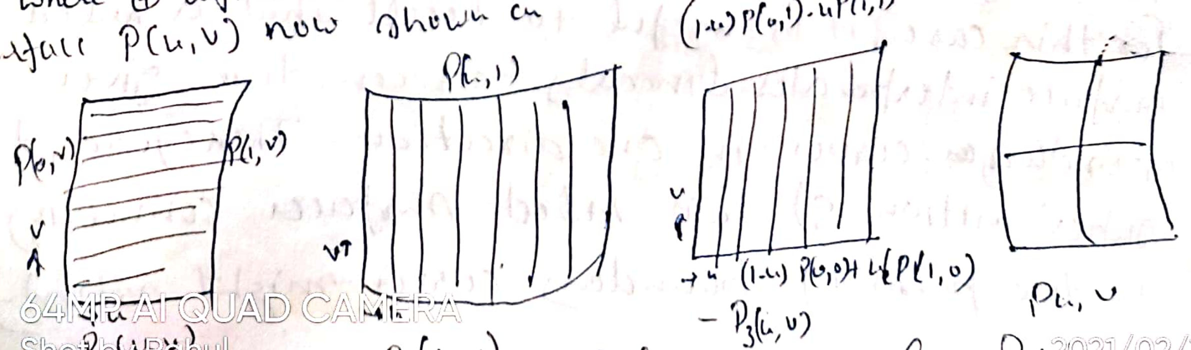
$P_3(u,v) = (1-v)[(1-u)P(0,0) + uP(1,0)] + v[(1-u)P(0,1) + u P(1,1)]$ (6)

Substituting (6) from (3)

$P(u,v) = P_1(u,v) + P_2(u,v) - P_3(u,v)$

or $P(u,v) = P_1(u,v) \oplus P_2(u,v)$

where $\oplus$ defines the "boolean sum" which is $P_1 + P_2 - P_3$. The surface $P(u,v)$ now shown as



Bilinearly Blended Coons Patch

Its matrix form is

$$P(u,v) = -\begin{bmatrix} -1 & (1-u) & u \end{bmatrix} \left[\begin{array}{c|cc} 0 & P(u,0) & P(u,1) \\ \hline P(0,v) & P(0,0) & P(0,1) \\ P(1,v) & P(1,0) & P(1,1) \end{array}\right] \begin{bmatrix} -1 \\ 1-v \\ v \end{bmatrix} \quad (7)$$

The left column & upper row of the matrix represent $P_1(u,v)$ & $P_2(u,v)$ respectively while the lower right block represent the correction surface $P_3(u,v)$. The functions $-1, 1-u, u, 1-v$ & v are called blending functions because they blend together four separate boundary curves to give one surface.

The main drawback of the bilinearly blended Coon patch is that it only provides C^0 continuity (positional continuity) between adjacent patches, even if their boundary curves form C^1 continuity network.

Gradient continuity across boundaries of patches of a composite coons surface is essential for practical applications. If, for example, a network of C^1 curves is given, it becomes very desirable to form a composite coon surface which is smooth or C^1 continuous, that is it provides continuity ~~across~~ of cross boundary derivative between patches. Eqn (7) shows that the choice of blending functions controls the behaviour of the resulting Coon patch. If the cubic Hermite polynomials $F_1(x)$ & $F_2(x)$ is given & are used instead of the linear polynomials $(1-u)$ & u respectively, a bicubically blended Coons patch results. This patch guarantee C^1 continuity between patches.

$$\therefore P_1(u,v) = (2u^3 - 3u^2 + 1)\, P(0,v) + (-2u^3 + 3u^2)\, P(1,v) \quad (8)$$

$$P_2(u,v) = (2v^3 - 3v^2 + 1)\, P(u,0) + (-2v^3 + 3v^2)\, P(u,1) \quad (9)$$

The matrix equation of the bicubic coon patch becomes

$$P(u,v) = -\begin{bmatrix} -1 & F_1(u) & F_2(u) \end{bmatrix} \begin{bmatrix} 0 & P(u,0) & P(u,1) \\ P(0,v) & P(0,0) & P(0,1) \\ P(1,v) & P(1,0) & P(1,1) \end{bmatrix} \begin{bmatrix} -1 \\ F_1(v) \\ F_2(v) \end{bmatrix}$$

5. <u>Offset Surface</u> :-

If an original patch & offset direction are given, the equation of the resulting offset patch can be written as

$$P(u,v)_{offset} = P(u,v) + \hat{n}(u,v)\, d(u,v)$$

where $P(u,v)$, $\hat{n}(u,v)$ & $d(u,v)$ are the original surface, the unit normal vector at point (u,v) on the original surface & the offset distance at point (u,v) on the original surface respectively. The unit normal $\hat{n}(u,v)$ is the offset direction. The distance $d(u,v)$ enables generating uniform or tapered thickness surfaces depending on whether $d(u,v)$ is constant or varies linearly in u, v or both.